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Jean-Marc Bottazzi[a] Jaime Luque[b] Mário R. Páscoa[c]

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Abstract

Securities markets theory includes repo and distinguishes shorting from issuing. Here we revisit whether trading alone can give rise to Ponzi schemes and rational bubbles. We show that under the same institutional arrangements that limit re-hypothecation (e.g., through segregated haircut rules or explicit leverage constraints on haircut collecting dealers), (1) trading Ponzi schemes are prevented without having to assume uniform impatience, (2) for securities in positive net supply, bubbles are ruled out under complete markets but may occur when markets are incomplete. We give an example of such a bubble, under a finite present value of wealth.

Keywords and Phrases : repo, short sale, bubble, repo specialness, Ponzi scheme, leverage.

JEL classification numbers : C62, D52, D53, D90, G12.

[a]Capula and Centre d'Economie de la Sorbonne, Paris School of Economics (CES), France.

[b]Departamento de Economia, Universidad Carlos III de Madrid, Spain.

[c]Corresponding author's address : Mário Páscoa, NOVA School of Business and Economics, Campus de Campolide, 1099-032 Lisboa, Portugal.

Email addresses : jean-marc@bottazzi.org (J.-M. Bottazzi), japluque@eco.uc3m.es (J. Luque), pascoa@fe.unl.pt (M. Páscoa).

1 Introduction

This paper revisits specifically for the securities market two major issues in infinite horizon equilibrium theory: the occurrence of Ponzi schemes and of price bubbles. We do this by capturing the basic fact that in order to short sell a security, one has to borrow it. In this context, shorting and issuing are perfectly distinguished. This allows us to focus on the pure trading aspects of securities. Rather than assuming solvency constraints or debt limits, we see how the way securities are traded already constrains the asymptotic values of security positions. Then, we address the problematic introduced by Santos and Woodford (1997), "the conditions under which asset prices in an inter-temporal competitive equilibrium are equal to" the expected discounted value of futures cash flows.

It is our view that repo market theory is a core part of security markets theory, as borrowing and lending of securities cannot be ignored. Models have to keep track of both security positions and (title) possession out of a finite pool. Formally, a repo trade consists in a security sale under the agreement of a future repurchase at a predetermined date and price. It is a collateralized loan, where the security that serves as a collateral is kept by the creditor (the borrower of the security), who tends to use it, lending on or shorting, before the delivery is due.

A bit out of necessity (so bubbles are not merely killed by the finite horizon!), we extend the literature on financial equilibrium that has incorporated the repo market in a general equilibrium model with finite horizon (see Duffie (1996) and Bottazzi, Luque and Páscoa (2011a)). Thus, we add a market for funding securities in an infinite horizon framework, absent in previous papers in the field (see Araujo, Páscoa and Torres-Martínez (2002), Hernández and Santos (1996), Magill and Quinzii (1994, 1996) and Santos and Woodford (1997)).

Also, we have seen how Bottazzi, Luque and Páscoa (2011a) introduces possession (or specialness) value as part of the normal value of a security. Thus, we include possession value into the fundamental value of a security. And we will say that there is a bubble in a given security if its value exceeds the sum of the expected discounted value of its cash flows plus its possession value.

When repo markets were ignored, the equilibrium literature had assumed two sorts of portfolio constraints, with quite different purposes. One, (a) a plain bounded short sales constraint, for equilibrium in finite horizon economies, in order to overcome Hart's (1975) problem. Another, (b) constraints on the value of debt (bounding it uniformly or relating it to future wealth) that managed to avoid Ponzi schemes, when uniform impatience was assumed. An exception was the model by Araujo, Páscoa and Torres-Martínez (2002) with collateralized promises, such as mortgages, where the same form of secured borrowing served both purposes[1] Obviously, the plain no-short-sales restriction also served both purposes, without any need for uniform impatience. However, it was hard to see what was the institutional framework behind constraints (a) or (b).

Knowing one needs to borrow a security in order to short it, there is a natural possession constraint, the non-negativity of title collected for each security, (in the "box"). Such non-negativity contraint is quite different from the above instrumental constraints.

In such a context, the two forms of limited re-hypothecation introduced in finite horizon (in Bottazzi, Luque and Páscoa (2011a)) readily extend to open ended horizon, where they actually also guarantee equilibrium even with infinite lived agents and without uniform impatience requirements. These are two institutional arrangements that end up restricting the way in which a borrowed security will be lent again several times (possibly by being shorted and then lent by the buyer[2]). One of these two forms is the provision that the security borrowed through repo cannot be fully shorted or lent, more precisely, the haircut portion of the pledged security, paid for with client money, cannot be re-used[3]. We call this the *direct limited re-hypothecation* case. The other form is based on

[1] The first purpose was met as the finite supply of asset to pledge as collateral in Asset Backed Securities (the housing stock for Mortgages) limits possible issuance. Interestingly, in the sub-prime crisis, when in 2007 Wall-street ran out of assets to pledge (sub-prime loan) to satiate the investor appetite, such a bound of the underlying stock of assets was stepped aside. The issuance went on artificially using the derivative markets through the synthetic CDOs market - see a narrative account of this in Michael Lewis's "The Big Short."). But when pledged assets are replaced by a combination of cash and CDS, the quantity of available asset is no upper bound on issuance. We think such arrangements should be discouraged in the future.

[2] Aitken and Singh (2009) addressed re-hypothecation in the narrower sense of lending a security that one had borrowed. Bottazzi, Luque and Páscoa (2011a) include also the re-hypothecation that occurs when the shorted security is lent by the buyer.

[3] This is a more and more common practice, and for the equivalent of haircuts in the derivatives market (initial margins), such initial margins are segregated when cleared on an exchange. Practice and regulation are more and more pushing towards such a segregation, after the Lehman crisis.

the current market practice of bounding the positions of dealers, who are exempt from paying haircut and actually collect haircuts from customer. We call this the *constrained dealers* case.

Limited re-hypothecation bounds the *leverage* that can be done in the economy. One may think that the haircut alone is enough to bound leverage (by the inverse of the haircut rate). This is true when leverage is done by just trading in one security and its respective repo market. But, when we combine these operations on more than one security, the haircut alone cannot bound leverage: the haircut paid in one security can be compensated by the haircut collected in another security. In this respect, our conclusion is that the institutional arrangements that bound leverage will also prevent Ponzi schemes.[4]

The equilibrium existence result has an important implication in terms of security pricing. A central result in the literature of asset pricing bubbles (see Santos and Woodford (1997) and Magill and Quinzii (1996)) asserts that, if short sales were allowed, the (previous) no-Ponzi schemes conditions (debt constraints coupled with uniform impatience) end up ruling out also bubbles, for deflators yielding finite present value of wealth and when assets were in positive net supply. In contrast, the two forms of limited re-hypothecation we introduced are now compatible with incomplete market bubbles for positive net supply securities, under time and state separable preferences.

We illustrate this result with two examples, one for each form of limited rehypothecation. There, consumers are impatient, although not uniformly (the discount factor is not stationary as beliefs change across different paths). This scenario allows for bubbles in positive net supply securities with associated repo market. The deflator is given by the marginal rates of substitution and yields a finite present value of wealth. This result adds to the examples by Santos and Woodford (1997) and Páscoa, Petrassi and Torres-Martinez (2011) that showed how an incomplete markets scenario without uniform impatience can generate bubbles of securities that cannot be shorted.

In the equilibria of these examples, there are no positive shadow prices for the box constraint and, therefore, the security price is just equal to the series of deflated future expected cash flows plus the bubble. If the box constraint had a shadow price, the security would be on *special* at that node.[5]

[4]In particular, if dealers were not constrained they could do a Ponzi scheme through the haircut collection process.

[5]More precisely, this is the case when the General Collateral (GC) rate would coincide with

In general, specialness and the bubble are compatible, at any node. These are now the two possible causes for the security price to exceed the series of discounted expected cash flows. There is an important reason why we include possession value in the fundamental value of the security: arbitrage should be defined by taking the box non-gnegativity constraint into account. We also show that, under the Kuhn-Tucker deflator process, a bubble is, nevertheless, incompatible with an asymptotic specialness (a positive difference between the repo and the general collateral rates, in the limit).

The next sections are as follows. In section 2 we present an infinite horizon model with repo markets. Section 3 studies individual optimality and provides equilibrium existence results, under the above two cases of limited rehypothecation. Section 4 addresses the possibility of doing Ponzi schemes by means of repo, when none of these two cases is being considered. Section 5 addresses bubbles and Section 6 provides two examples of incomplete markets equilibrium with bubbles in a positive net supply security. Section 7 discusses specialness in the infinite horizon setting.

2 The infinite horizon model

2.1. *Uncertainty*

We consider a discrete time infinite horizon economy. The set of dates is given by $T = \{0, 1, ...\}$. The initial date is free of uncertainty, whereas at each of the following dates a finite set of states of nature may occur. An information set $\xi = (t, \bar{s}_t, s)$ is called a node of the economy, where $t \in T$ represents the date, $\bar{s}_t = (s_0, ..., s_{t-1})$ the previous history of realizations of states of nature, and s the state that occurs at date t. We say that $\mu = (t', \bar{s}_{t'}, s')$ is a successor of ξ, and write $\mu \geq \xi$ if $t' \geq t$ and $\bar{s}_{t'} = (\bar{s}_t, s, ...)$. The set of immediate successors of node ξ is denoted by ξ^+. The unique predecessor of ξ is denoted by ξ^-. The unique information set at $t = 0$ is ξ_0. We write $\mu > \xi$ if $\mu \geq \xi$ but $\mu \neq \xi$.

The set of nodes is denoted by D and it is called the event-tree. Let $D(\xi) = \{\mu \in D : \mu \geq \xi\}$ be the subtree with root ξ. The set of nodes with date T in $D(\xi)$ is denoted by $D_T(\xi)$. Finally, let $D^T(\xi) = \cup_{k=t(\xi)}^{T} D_T(\xi)$ be the set of successors

the interest rate on unsecured borrowing (say, the Fed funds rate). Otherwise, if the GC rate were itself below the unsecured rate, a box shadow price might occur without the security being on special (see Bottazzi, Luque and Páscoa (2011b)). See also Duffie (1996)) on specialness.

of ξ with date less or equal than T.

2.2. *Infinite lived agents and commodities*

There is a finite set of infinite lived agents $\mathbf{I}$. There are L different types of commodities that agents use for consumption. Each agent $i \in \mathbf{I}$ has an endowment of commodities $\omega^i \in \mathbb{R}_{++}^{L\times D}$, which is assumed to be bounded from below at each node. Total endowments at node ξ are denoted by $\Omega_\xi \equiv \sum_i \omega_\xi^i$. The consumption set is $\mathbb{R}_{+}^{L\times D}$. Agent i's utility function, given by $U^i : \mathbb{R}_{+}^{L\times D} \to \mathbb{R}_+ \cup \{+\infty\}$, is assumed to be separable in time and states of nature, i.e., for any $x \in \mathbb{R}_{++}^{L\times D}$, we define $U^i(x) \equiv \sum_{\xi \in D} u_\xi^i(x_\xi)$, where $u_\xi^i(.)$ is the Bernoulli utility function at node ξ. Moreover, we also impose the following standard assumptions:

(A1) (i) For any node $\xi \in D$, the function u_ξ^i is twice continuously differentiable and strictly increasing; (ii) $u_\xi^i(0) = 0$ and $\sum_{\xi\in D} u_\xi^i(\Omega_\xi) < +\infty$; (iii) $\mathcal{D}u_\xi^i(x_\xi) \in \mathbb{R}_{++}^L$, $\forall x_\xi \in \mathbb{R}_+^L$; (iv) $\forall c \in \mathbb{R}$, the set $[u_\xi^i]^{-1}(c)$ is closed in $\mathbb{R}_{++}^L$, and (v) at every $x_\xi \in \mathbb{R}_{++}^L$, $h' \cdot \mathcal{D}^2 u_\xi^i(x_\xi) \cdot h < 0$, $\forall h \neq 0$.[6]

(A2) Commodity endowments are uniformly bounded away from zero: for any $i \in \mathbf{I}$, $\omega^i \ggg 0$.

2.3. *Security markets*

We conceive the initial period ξ_0 as a situation where issuance has already happened, and issued securities have been placed. Each agent i thus has initial endowments $e^i \in \mathbb{R}_{++}^J$ of securities, describing his holdings when trading starts.

Securities are traded at every node in the event-tree. We denote a *trade* in security j at node $\xi \in D$ by $y_{j\xi}^i$. Agent i's security j *position* at node ξ is $\varphi_{j\xi}^i$. At the initial node ξ_0 the position is $\varphi_{j\xi_0}^i = e_j^i + y_{j\xi_0}^i$. For node $\xi > \xi_0$, the corresponding position is $\varphi_{j\xi}^i = \varphi_{j\xi^-}^i + y_{j\xi}^i$ (the previous position $\varphi_{j\xi^-}^i$ plus current trade $y_{j\xi}^i$). A *short sale* at node $\xi \in D$ occurs when the position gets negative, that is, when $\varphi_{j\xi}^i < 0$.

We model securities as real assets. The case of securities with exogenous nominal yields could be easily accommodated. The real proceeds of security j

[6]Here $\mathcal{D}$ denotes the differential operator. Items (i), (iii) and (iv) were already assumed in Bottazzi, Luque and Páscoa (2011a). Item (v) already appeared but in the weaker form of differential strict quasi-concavity. These items allow us to bound the intertemporal rates of substitution from above and from below. Inada conditions would be too extreme in this model with repo markets, where a period can be very short, possibly overnight.

at node $\xi > 0$ are given by a non-zero vector $B_{j\xi} \in \mathbb{R}^L_+$. Moreover, we assume that each good has at least some security paying in that good.[7] Formally, we assume that for any $\xi \in D$, the real returns matrix B_ξ of type $L \times J$ does not have null rows. Given spot prices at node $\xi \in D$, $p_\xi \in \mathbb{R}^L_+$, the nominal return of security j is then given by $p_\xi B_{j\xi}$. The security market transactions of node ξ take place at a price denoted by $q_{j\xi}$. Notice that this characterization of the returns matrix actually is in accordance with the common view of understanding securities as valuables - a security can be thought to represent the value of a commodity basket.

Taking into account security proceeds, we have that the total resources of physical commodities at node $\xi \in D$ are $\sum_i \tilde{\omega}^i_\xi \equiv \sum_i \omega^i_\xi + \sum_j B_{j\xi} \sum_i \varphi^i_{j\xi^-}$

2.4. *Repo markets*

We introduce repo trading by using the variable z. Notation follows Bottazzi, Luque and Páscoa (2011). An agent is said to be *long* in repo, denoted by $z > 0$, if he borrows the security in exchange of a cash loan. On the other hand, the agent is *short* in repo, denoted by $z < 0$, if he is the lender of the security and the borrower of cash. Repos are traded at every node $\xi \in D$. The loan associated with repo is $\pi_{j\xi} z_{j\xi}$, where $z_{j\xi}$ represents the amount of security j engaged in the repo and $\pi_{j\xi}$ is the haircutted price of the collateralized loan signed at node ξ. The haircut $(1 - h_{j\xi}) \in [0, 1]$, exogenously given in this model, is imposed to compensate the lender of funds with the risk associated with a simultaneous default and adverse market move of the security lent.[8] For the sake of simplicity and following typical market practice, we assume that all repos on the same security and same node $\xi \in D$ share a common haircut.[9] The collateralized loan price is then $\pi_{j\xi} = h_{j\xi} q_{j\xi}$. We assume that, for a repo contract signed at node ξ^-, the repurchase takes place at the following date (that is, at $t(\xi^-) + 1$) at repo interests $r_{j\xi}$. The repo rate (or interest rate) on a repo loan at node ξ^- is $r_{j\xi} - 1 \equiv \rho_{j\xi}$.

Securities borrowed can be re-hypothecated, that is, lent or short sold (and

[7] For example, if there is a forward term contract for each good.

[8] For endogenous haircuts in the case of mortgages see Geanakoplos [14] and Araujo, Fajardo and Páscoa [3]. In Fostel and Geanakoplos [13] the margins on financial assets collateralizing money promises are also endogenous. In a recent paper, Brunnermeier and Pedersen [7] address the dependence of margins or haircuts on asset's market liquidity.

[9] This can and should be relaxed when we focus more on credit of the trading entities - something we do not go into here.

then lent by the buyer). Bottazzi, Luque and Páscoa (2011a) showed that, for economies with more than one security, existence of equilibrium in a two-dates economy requires the rehypothecation of securities to be bounded. Two scenarios were shown to verify this: direct limited rehypothecation and constrained dealers. These scenarios are now adapted to our infinite horizon model.

I) Direct limited re-hypothecation

In this case the security borrower cannot re-hypothecate a certain percentage of the borrowed amount of the security. To make a simple argument we assume that haircut paid is being segregated.[10] Thus, the re-hypothecation percentage should be no more than 1 - the haircut rate, as the excess security pledged is not lent on. That is, the fraction $H_{j\xi}$ of a security j that can be sold or lent out in repo after being borrowed at node ξ must satisfy $H_{j\xi} \leq h_{j\xi} < 1$. The haircut posted by counterparties (and paid for with their own funds) is set aside.

Budget constraints in this infinite horizon framework depart from previous ones in the literature. Agent i's budget constraints at nodes ξ_0 and $\xi > \xi_0$ are, respectively,

$$g^i_{\xi_0}(a_{\xi_0}; p, q, r) \equiv p_{\xi_0}(x_{\xi_0} - \omega^i_{\xi_0}) + q_{\xi_0}(\varphi_{\xi_0} - e^i + h_{\xi_0} z_{\xi_0}) \leq 0 \tag{1}$$

$$g^i_\xi(a_\xi, a_{\xi^-}; p, q, r) \equiv p_\xi(x_\xi - \omega^i_\xi) + q_\xi(\varphi_\xi - \varphi_{\xi^-} + h_\xi z_\xi) - p_\xi B_\xi \varphi_{\xi^-} - q_{\xi^-} r_{\xi^-} h_{\xi^-} z_{\xi^-}) \leq 0 \tag{2}$$

where $a_\xi \equiv (x_\xi, \varphi_\xi, z_\xi)$ denotes an agent's plan at node ξ.[11] This budget constraint says that the value of the excess of commodities demand, asset purchases and repo long positions ($z_\xi > 0$) minus short positions ($z_\xi < 0$) must be below the commodity returns of accumulated asset positions (φ_{ξ^-}) and the repo loan repayments of the previous repo agreements ($q_{\xi^-} r_{\xi^-} h_{\xi^-} z_{\xi^-}$).

Agent i's box constraint of security j at node ξ is

$$f^{i,H}_{j\xi}(a_\xi) \equiv \varphi_{j\xi} + H_\xi z^+_{j\xi} - z^-_{j\xi} \geq 0 \tag{3}$$

[10] There is an evolution of practice in that direction, it may be accentuated by regulation, including interpretation of existing law (as haircut is paid for with customer money). Note how this assumption is not dissimilar to the segregation of initial margin by exchanges for vanilla derivatives like interest rate swaps.

[11] For notation brevety, when two vectors $a = (a_1, ..., a_N)$ and $b = (b_1, ..., b_N)$, with the same dimension, appear multiplied, ab, we mean the vector $a \Box b = (a_1 b_1, ..., a_N b_N)$. This is the case of products $h_\xi z_\xi$ and $r_\xi h_{\xi^-} z_{\xi^-}$.

where $z^+_{j\xi} = \max\{0, z_{j\xi}\}$ and $z^-_{j\xi} = -\min\{0, z_{j\xi}\}$. So, $z_{j\xi} = z^+_{j\xi} - z^-_{j\xi}$. Now, $f^{i,H}_{j\xi}$ is a concave function as $f^{i,H}_{j\xi}(a_\xi) = \varphi_{j\xi} + z_{j\xi} - (1 - H_\xi)z^+_{j\xi}$.

II) Constrained dealers

As it currently happens, not all agents get the cash benefit from collecting a haircut when borrowing a security - in other words, giving a cash loan and receiving a security as collateral that is worth more. Agents whose business is intermediation (Dealers/Prime brokers) have this cash benefit, but in practice have their positions bounded in value by regulation. The mechanisms often are among other things BIS ratios limits. Their customers (e.g., hedge funds, mutual funds, retail securities brokers, private banks and insurance companies) do not face such regulation on their positions, but must pay haircut when lending securities. We refer to the former as dealers (**D**) and to the latter as non-dealers (**ND**). As it usually occurs, we assume that non-dealers only engage in repo with dealers.

Due to the asymmetry in the haircut treatment, we have to use different variables for security borrowing and lending. We denote the long and short repo positions at node ξ by $\theta_\xi \geq 0$ and $\psi_\xi \geq 0$, respectively. When a dealer is long in repo (and thus the non-dealer is short) the interest rate that applies is $r_{1\xi}$, while if the dealer is short in repo (and thus the non-dealer is long) the interest rate that applies is $r_{2\xi}$. Notice that dealers like to take positions in these two variables, whereas non-dealers prefer to have just one of them non-null (see Bottazzi, Luque and Páscoa (2011a)). However, regulation prevents dealers from taking positions in the two variables that are too big.

Dealer $i \in \mathbf{D}$ budget constraints at nodes ξ_0 and $\xi > \xi_0$ are, respectively,

$$g^i_{\xi_0}(a_{\xi_0}; p, q, r) \equiv p_{\xi_0}(x_{\xi_0} - \omega^i_{\xi_0}) + q_{\xi_0}(\varphi_{\xi_0} - e^i + h_{\xi_0}\theta_{\xi_0} - \psi_{\xi_0}) \leq 0 \tag{4}$$

$$\begin{aligned} g^i_\xi(a_\xi, a_{\xi^-}; p, q, r) \equiv\ & p_\xi(x_\xi - \omega^i_\xi) + q_\xi(\varphi_\xi - \varphi_{\xi^-} + h_\xi\theta_\xi - \psi_\xi)- \\ & - p_\xi B_\xi \varphi_{\xi^-} - q_{\xi^-}(r_{1\xi^-}h_{\xi^-}\theta_{\xi^-} - r_{2\xi^-}\psi_{\xi^-}) \leq 0 \end{aligned} \tag{5}$$

Non-dealer $i \in \mathbf{ND}$ budget constraints at nodes ξ_0 and $\xi > \xi_0$ are, respectively,

$$g^i_{\xi_0}(a_{\xi_0}; p, q, r) \equiv p_{\xi_0}(x_{\xi_0} - \omega^i_{\xi_0}) + q_{\xi_0}(\varphi_{\xi_0} - e^i + \theta_{\xi_0} - h_{\xi_0}\psi_{\xi_0}) \leq 0 \tag{6}$$

$$\begin{aligned} g^i_\xi(a_\xi, a_{\xi^-}; p, q, r) \equiv\ & p_\xi(x_\xi - \omega^i_\xi) + q_\xi(\varphi_\xi - \varphi_{\xi^-} + \theta_\xi - h_\xi\psi_\xi)- \\ & - p_\xi B_\xi \varphi_{\xi^-} - q_{\xi^-}(r_{2\xi^-}\theta_{\xi^-} - r_{1\xi^-}h_{\xi^-}\psi_{\xi^-}) \leq 0 \end{aligned} \tag{7}$$

For both dealers and non-dealers, the security j box constraint of an agent $i \in \mathbf{D} \cup \mathbf{ND}$ at node ξ is

$$f^i_{j\xi}(a_\xi) \equiv \varphi_{j\xi} + \theta_{j\xi} - \psi_{j\xi} \geq 0 \tag{8}$$

We assume that the borrowing of securities by dealers is bounded by regulation in the following way:

(A3) The real values of dealer i's long repo positions are uniformly bounded, i.e., for each j,

$$\frac{q_{j\xi}}{\sum_l p_{l\xi}} \theta_{j\xi} \leq M_j \tag{9}$$

As the regulator sets upper bounds on real rather than just nominal values, this policy does not suffer from monetary illusion. That is, the regulator takes into account inflation (or deflation) when setting the bounds.

Notice that (A3) implies that the real value of dealers' short sales is also uniformly bounded. In fact, the box constraint together with (A3) imply $\frac{q_{j\xi}}{\sum_l p_{l\xi}} \varphi_{j\xi} \geq -M_j$. Notice that the weaker version holds: nominal values of security borrowing and short-sales are clearly (uniformly) bounded (as (p_ξ, q_ξ) can be normalized to be in the simplex). This implies that *feasible* security (and repo positions) have (uniformly) bounded nominal values, which is what Bottazzi, Luque and Páscoa (2011a) assumed to show existence of equilibrium in a finite horizon economy of type II.

3 Individual optimality and equilibrium

We now introduce the equilibrium concept for this economy. Let us first denote by a^i a plan for agent i at (p, q, r). For case I (direct limited rehypothecation) an individual plan is given by the vector $a^i = (x^i, \varphi^i, z^i) \in \mathbb{R}^{L\times D}_+ \times \mathbb{R}^{J\times D} \times \mathbb{R}^{J\times D}$, where x^i is subject to the sign constraint $x^i \geq 0$. For case II (constrained dealers) this plan is $a^i = (x^i, \varphi^i, \theta^i, \psi^i) \in \mathbb{R}^{L\times D}_+ \times \mathbb{R}^{J\times D} \times \mathbb{R}^{J\times D}_+ \times \mathbb{R}^{J\times D}_+$, where $x^i \geq 0$, $\theta^i \geq 0$ and $\psi^i \geq 0$. We say that a plan a^i is *individually admissible* for agent i if it satisfies constraints (2) and (3) in case I, (5), (8) and (9) in case II with i being a dealer, and (7) and (8) if case II with i being a non-dealer. Consumer i's problem is said to be optimal if he chooses an admissible plan a^i that maximizes his utility U^i. We assume that

Definition 1: *An equilibrium for this economy consists on a vector of prices* $(p, q, r) \in \mathbb{R}_+^{L\times D} \times \mathbb{R}_+^{J\times D} \times \mathbb{R}_+^{J\times D}$ *and individual plans* $(\bar{a}^i)_{i\in\mathbf{I}}$*, such that,*

(i) for each agent $i \in \mathbf{I}$*, the plan* $\bar{a}^i$ *is optimal at prices* (p, q, r)*.*

(ii) at any node $\xi \in D$*, commodity, security and repo markets clear, that is,*

$$\sum_{i\in\mathbf{I}} \bar{x}^i_\xi - \Omega_\xi = 0; \quad \sum_{i\in\mathbf{I}} (\bar{\varphi}^i_\xi - e^i) = 0;$$

and

$$\begin{aligned} \sum_{i\in\mathbf{I}} \bar{z}^i_\xi &= 0, \text{ if case I} \\ \sum_{i\in\mathbf{ND}} \bar{\theta}^i_\xi &= \sum_{i\in\mathbf{D}} \bar{\psi}^i_\xi = 0 \text{ and } \sum_{i\in\mathbf{D}} \bar{\theta}^i_\xi = \sum_{i\in\mathbf{ND}} \bar{\psi}^i_\xi = 0, \text{ if case II} \end{aligned}$$

Before characterizing the Euler and transversality conditions for this economy, let us first denote agent i's Lagrangian at node ξ by[12]

$$L^i_\xi(a_\xi, a_{\xi^-}; \lambda^i_\xi, \mu^i_\xi, p, q, r) \equiv u^i_\xi(x_\xi) - \lambda^i_\xi g^i_\xi(a_\xi, a_{\xi^-}; p, q, r) + \mu^i_\xi f^i_\xi(a_\xi)$$

Let $L^i_{1\xi}$ and $L^i_{2\xi}$ denote super-gradient vectors of L^i_ξ with respect to present and previous plans a_ξ and a_{ξ^-}, respectively.

Proposition 1: *If* $\bar{a}^i$ *is an optimal solution to (i) of Definition 1, then there exist multipliers* $(\lambda^i_\xi)_{\xi\in D} \gg 0$ *and* $(\mu^i_\xi)_{\xi\in D} \geq 0$ *together with super-gradients for* u^i_ξ *and* f^i_ξ *at* $\bar{a}^i_\xi$ *such that the following Euler conditions are satisfied*

$$L^i_{1\xi}(\bar{a}^i) + \sum_{\eta\in\xi^+} L^i_{2\eta}(\bar{a}^i) \leq 0 \tag{10}$$

$$\left(L^i_{1\xi}(\bar{a}^i) + \sum_{\eta\in\xi^+} L^i_{2\eta}(\bar{a}^i) \right) \bar{a}^i_\xi = 0 \tag{11}$$

and the following transversality condition holds

$$\limsup_{T\to\infty} \sum_{\xi:t(\xi)=T} \left(\sum_{\eta\in\xi^+} L^i_{2\eta}(\bar{a}^i)\bar{a}^i_\xi \right) \leq 0 \tag{12}$$

[12] This is the Lagrangian of a convex problem and we refer the reader to Rockafellar (1970).

Proposition 2: *Let $\bar{a}^i$ be an admissible plan for agent i, at prices (p, q, r). Suppose $\bar{a}^i$ satisfies Euler and transversality conditions (10), (11) and (12), respectively, at (p, q, r). Then, $\bar{a}^i$ is individually optimal if, for any plan a^i satisfying the corresponding individual's budget and box constraints*[13], *we have*

$$\limsup_{T\to\infty} \sum_{\xi:t(\xi)=T} L^i_{1\xi}(\bar{a}^i)a^i_\xi \leq 0 \tag{13}$$

Inequality (13) means that the plan a, of consumption, security and repo trades, when weighted by current marginal benefits, has a vanishing value. In a model where naked short-sales are naively allowed, (13) reduces to the well-known condition that the plan a should not allow the agent to be a debtor at infinity ($\liminf_T \sum_{\xi:t(\xi)=T} \lambda_\xi q_{j\xi}\varphi_{j\xi} \geq 0$). Now the current marginal benefit must take into account the cost of borrowing securities (that is, $\lambda_\xi q_{j\xi}$ would be reduced by $\mu_{j\xi}$). See Section 4 on repo Ponzi schemes for a detailed analysis.

Santos and Woodford (1997) showed that an exogenous restriction to debt is equivalent to restriction (13) if the strong assumption of uniform impatience is imposed. In this paper we do not need to impose uniform impatience, as our natural modeling of short sales for securities does not need it.

Lemma 1: *The present value of wealth is finite for the Lagrange multipliers deflator process λ^i.*

The proof of Lemma 1 is established in the Appendix, as in Proposition 2 in Páscoa, Petrassi and Torres-Martínez (2011). Now, let us show that:

Proposition 3: *Let $\bar{a}^i$ be an admissible plan for agent i, at prices (p, q, r), satisfying Euler conditions* (10) *and* (11), *in cases I or II. Then, the condition* (13) *holds (in both cases I or II, respectively).*

We get the following existence result:

Theorem 1: *Equilibrium exists, in case (I) under (A.1) and in case (II) under (A.1), (A.2) and (A.3).*

The proof, given in the Appendix, uses Propositions 2 and 3.

[13] Constraints (2) and (3) in case I, (5), (8) and (9) in case II with i being a dealer, and (7) and (8) if case II with i being a non-dealer.

4 Repo Ponzi schemes

Outside of cases I or II, Ponzi schemes can be done, envolving a new type of infinite horizon arbitrage combining security and repo positions. Suppose budget constraints are, as in case I, described by (2) but limited rehypothecation is not imposed, so that the security j box constraint is as follows, at each node ξ:

$$\varphi_{j\xi} + z_{j\xi} \geq 0 \tag{14}$$

To simplify, we assume the economy to be deterministic, although the argument extends to stochastic economies straighforwardly. Now, given any plan (x, φ, z) satisfying budget and box constraints ((2) and (14), respectively), for a consumer i, we reduce the security position φ_t and match this by increasing the repo position z_t in the same amount at some date t, with a net gain in income (due to the haircut) and then repeat this procedure at all following dates, but possibly with different amounts in order to accommodate the changes in debt and dividends. Such variation can then be scaled up arbitrarily.

To be more precise, we consider the following variation: at date t, the repo position z_t is increased by $\varepsilon_t > 0$ and the security position φ_t is decreased by ε_t. Box constraint (14) remains satisfied at date t, but this joint operation results in a gain at date t given by $q_t(1-h_t)\varepsilon_t$ that can be spent on extra consumption. At the following date, $t+1$, the consumer can accommodate the variation in dividends and debt (net of the settlement of the repo variation) by finding $\varepsilon_{t+1} > 0$, so that z_{t+1} is increased by ε_{t+1}, φ_t is decreased by $\varepsilon_t > 0$ and φ_{t+1} is decreased by ε_{t+1}, while preserving the budget constraint (2) at date $t+1$. That is, ε_{t+1} must satisfy

$$q_{t+1}(1 - h_{t+1})\varepsilon_{t+1} - (q_{t+1} + p_{t+1}B_{t+1})\varepsilon_t + q_t h_t r_t \varepsilon_t \geq 0$$

Hence, we must have

$$\varepsilon_{t+1} \geq \frac{(q_{t+1} + p_{t+1}B_{t+1} - q_t h_t r_t)\varepsilon_t}{q_{t+1}(1 - h_{t+1})}$$

This condition can always be satisfied by any $\varepsilon_{t+1} > 0$ when the numerator on the right hand side is negative and by $\varepsilon_{t+1} > 0$ large enough if the numerator is positive, which would be the case when Euler equations on security and repo positions hold[14]. We repeat the procedure at the following dates and obtain a

[14]These conditions hold in the form of the following equations (as the variables are not sign

vector of increments $(\varepsilon_t, \varepsilon_{t+1}, ...)$ that determines an increase in the utility of consumer i. By multiplying this vector by an arbitrarily large scalar $\alpha > 0$ we get unbounded utility gains.

Analogous Ponzi scheme could not be done in case I or II. In case I, if $z_t > 0$, for box constraint (3) to hold, when z_t increases by ε_t, φ_t cannot decrease more than $H_t \varepsilon_t$. However, as $H_t \leq h_t$, the combination of these variations involves a cost to the consumer, at least equal to the extra loan given in repo $(q_t h_t \varepsilon_t)$ minus the decrease in the security position $(q_t H_t \varepsilon_t)$. If $z_t < 0$, although it would possible to decrease both z_t^- and φ_t by ε_t, with a resulting income gain $(q_t(1-h_t)\varepsilon_t)$, such arbitrage could not be scaled up by an arbitrarily high $\alpha > 0$, as z_t would become positive.

In case II, the box constraint (8) allows for an increment ε_t in θ_t accompanied by a reduction ε_t in φ_t. If the consumer is a non-dealer, there would be no change in income at date t (as no haircut is being collected by him, see (7)). However, this joint operation yields a non-positive payoff at the next date, given by $q_t r_t \varepsilon_t - (q_{t+1} + p_{t+1} B_{t+1})\varepsilon_t$. In fact, this difference is non-positive, as we know that Euler conditions hold (since an optimum was shown to exist, by Proposition 3), which tell us that $q_{t+1} + p_{t+1} B_{t+1} = \frac{\lambda_t^i}{\lambda_{t+1}^i} q_t - \frac{\mu_t^i}{\lambda_{t+1}^i}$ and $q_t r_t \leq \frac{\lambda_t^i}{\lambda_{t+1}^i} q_t - \frac{\mu_t^i}{\lambda_{t+1}^i}$ (the inequality form is due to the sign constraint on security borrowing). This cost cannot be compensated by repeating the operation at $t+1$, since again an increment in θ_{t+1} accompanied by a symmetric reduction in φ_{t+1} would not bring an income gain (by the same argument using (7)).

If the consumer were a dealer instead, the above joint modification of θ_t and φ_t would result in an income gain $(1-h_t)q_t\varepsilon_t$ at date t (according to (5)). However, dealers have bounded positions and, therefore, such arbitrage could not be scaled up arbitrarily. This shows the importance of regulations around balance-sheet limitation and haircut segregation for dealers.

constrained) $q_{t+1} + p_{t+1} B_{t+1} = \frac{\lambda_t^i}{\lambda_{t+1}^i} q_t - \frac{\mu_t^i}{\lambda_{t+1}^i}$ and $q_t h_t r_t = \frac{\lambda_t^i}{\lambda_{t+1}^i} q_t h_t - \frac{\mu_t^i}{\lambda_{t+1}^i}$.

5 Bubbles in complete markets

Using the Euler conditions on security positions, recursively, we obtain the following pricing formula for security j at node η

$$q_{j\eta} = \overbrace{\underbrace{\sum_{\xi>\eta} \frac{\lambda^i_\xi}{\lambda^i_\eta} p_\xi B_{j\xi}}_{\text{discounted dividends}} + \underbrace{\sum_{\xi\geq\eta} \frac{\mu^i_{j\xi}}{\lambda^i_\eta}}_{\text{specialness}}}^{\text{fundamental security value}} + \underbrace{\frac{1}{\lambda^i_\eta} \lim_T \sum_{\xi\geq\eta:t(\xi)=T} \lambda^i_\xi q_\xi}_{\text{bubble}} \tag{15}$$

where the sum of the two series constitutes the fundamental value and the last term is the bubble.

Proposition 4: *Bubbles cannot occur under complete markets.*

Observe that the absence of bubbles refers to the unique deflator, given by common marginal rates of intertemporal substitution, for which the present value of wealth is finite (see Lemma 1).

Proof of Proposition 4: To see this, recall that the transversality condition is

$$\limsup_T \sum_{\xi:t(\xi)=T} \left(-L^i_{1\xi}(\bar{a}^i)\cdot \bar{a}^i_\xi\right) \leq 0 \tag{16}$$

In case I we write (16) as

$$\limsup_T \sum_{\xi:t(\xi)=T} (-\mathcal{D}u^i_\xi + \lambda^i_\xi p_\xi)\bar{x}^i_\xi +$$
$$+ (\lambda^i_\xi q_\xi - \mu^i_\xi)\bar{\varphi}^i_\xi + (\lambda^i_\xi q_\xi h_\xi - \mu^i_\xi \chi_\xi)\bar{z}^i_\xi \leq 0$$

Now, $\mu^i_\xi(\bar{\varphi}^i_\xi + \chi_\xi \bar{z}^i_\xi) = 0$ implies[15]

$$\limsup_T \sum_{\xi:t(\xi)=T} \lambda^i_\xi(q_\xi\bar{\varphi}^i_\xi + q_\xi h_\xi \bar{z}^i_\xi) \leq$$
$$\leq 0 + \limsup_T \sum_{\xi:t(\xi)=T} \lambda^i_\xi(\mathcal{D}u^i_\xi - \lambda^i_\xi p_\xi)\bar{x}^i_\xi \leq 0$$

If $\lambda^i_\xi = \lambda_\xi, \forall i$, then

$$\limsup_T \sum_{\xi:t(\xi)=T} \lambda_\xi q_\xi (\sum_i \bar{\varphi}^i_\xi + h_\xi \sum_i \bar{z}^i_\xi) \leq$$
$$\leq \sum_i \limsup_T \sum_{\xi:t(\xi)=T} \lambda_\xi q_\xi(\bar{\varphi}^i_\xi + h_\xi \bar{z}^i_\xi) \leq 0$$

[15] Recall that $\limsup(A-B) \geq \limsup(A) - \limsup(B)$.

Hence, we have $\limsup_T \sum_{\xi:t(\xi)=T} \lambda_\xi q_\xi e \leq 0$, implying that $\limsup_T \sum_{\xi:t(\xi)=T} \lambda_\xi q_\xi \leq 0$. At any node η, $0 \leq \lim_T \sum_{\xi \geq \eta:t(\xi)=T} \lambda_\xi q_\xi \leq \lambda_\eta q_\eta < \infty$ and

$$\lim_T \sum_{\xi \geq \eta:t(\xi)=T} \lambda_\xi q_\xi \leq \limsup_T \sum_{\xi:t(\xi)=T} \lambda_\xi q_\xi$$

so $\lim_T \sum_{\xi \geq \eta:t(\xi)=T} \lambda_\xi q_\xi = 0$.

Now we show that bubbles cannot occur either in case II under complete markets. In fact, in this case (16) implies that

$$\limsup_T \sum_{\xi:t(\xi)=T} \lambda_\xi q_\xi (\sum_i \bar{\varphi}^i_\xi + h_\xi (\sum_d \theta^d_\xi - \sum_{nd} \psi^{nd}_\xi) + \\ + (\sum_{nd} \theta^{nd}_\xi - \sum_d \psi^d_\xi)) \leq 0$$

Market clearing then implies $\limsup_T \sum_{\xi:t(\xi)=T} \lambda_\xi q_\xi e \leq 0$ and, therefore, for any node η, $\lim_T \sum_{\xi \geq \eta:t(\xi)=T} \lambda_\xi q_\xi = 0$. ■

However, under incomplete markets the term $\lim_T \sum_{\xi \geq \eta:t(\xi)=T} \lambda_\xi q_\xi$ may be positive, as shown in the examples below.

6 Bubbles in incomplete markets: Examples

We give now two examples where bubbles occur, under a deflator yielding finite present value of wealth, for a positive net supply asset that is shorted in equilibrium. Preferences and endowments do not satisfy uniform impatience conditions, but these are not required for existence of equilibrium in cases I or II. Recall that in earlier models, equilibrium existed when short sales were either ruled out or were treated as naked short sales (although subject to constraints on the value of borrowing), but uniform impatience had to be assumed in the latter.

Example 1: This is an example of a security and repo equilibrium for an economy with two infinite lived agents, A and B, trading one commodity and one security in sequential incomplete markets. Ponzi schemes are avoided by limited rehypothecation (case I, with $h = H$) and the security has a price bubble. Preferences and endowments are adapted from an example of a monetary equilibrium in Páscoa, Petrassi and Torres-Martínez (2011), but fiat money (with a no-short-sales constraint) is now replaced by a security paying real dividends.

Portfolios must satisfy the box constraint (3): security purchases can be funded and short sales can be done using what has been borrowed of the security.

The infinite tree D has two branches, up ($\xi_{\bar{s}_t u} \equiv (t, \bar{s}_t, u)$) or down ($\xi_{\bar{s}_t d} \equiv (t, \bar{s}_t, d)$), at each node ξ. We denote by $\xi_{\bar{s}_t u}$ the node attained after the history of node realizations $\bar{s}_t$ by going up (and similarly for $\xi_{\bar{s}_t d}$). Preferences are given by $U^i(x) = \sum_{\xi \in D} \beta^{t(\xi)} \rho^i_\xi x_\xi$, for $i = A, B$, where $\beta \in (0,1)$ is the discount factor and $\rho^i_{\xi_{\bar{s}_t}} \in (0,1)$ is the probability belief at node $\xi_{\bar{s}_t}$ satisfying $\rho^i_{\xi_0} = 1$, $\rho^i_{\xi_{\bar{s}_t}} = \rho^i_{\xi_{\bar{s}_t u}} + \rho^i_{\xi_{\bar{s}_t d}}$, $\rho^A_{\xi_u} = (1/2^{t(\xi)+1})\rho^A_{\xi_{\bar{s}_t}}$ and $\rho^B_{\xi_{\bar{s}_t u}} = (1 - (1/2^{t(\xi)+1}))\rho^B_{\xi_{\bar{s}_t}}$. We denote by $\xi_{\bar{s}_t ud}$ the node attained after the history of node realizations $\bar{s}_t$ by going up and then down (and similarly for other pairs of branches).

Commodity endowments have a trend component equal to 1 for both agents. Endowment shocks benefit agent A when down is followed by up, while agent B gets also a positive shock but when up is followed by down and at ξ_{0d}. More precisely, agent A's endowment is $\omega^A_\xi = 1 + P_{t(\xi)}$ if $\xi = \sigma_{du}$ for some $\sigma \in D$ and equals 1 otherwise. Agent B's endowment is $\omega^B_\xi = 1 + P_{t(\xi)}$ if $\xi = \xi_{0d}$ or $\xi = \sigma_{ud}$ for some $\sigma \in D$ and equals 1 otherwise. The positive shocks $P_{t(\xi)}$ will be specified below.

The security pays B_t units of the commodity at every node occurring at date t.[16] Agent A is endowed with 1 unit of the security at ξ_0, while agent B is not endowed.

Equilibrium prices: Let us take the security to be the numeraire and denote the commodity price by p_ξ. We will find equilibrium prices that depend only on the date and denote these by $p_{t(\xi)}$. Euler conditions on consumption are satisfied when the Lagrange budget multiplier λ^i_ξ is given by $\lambda^i_\xi = \rho^i_\xi \beta^{t(\xi)}/p_\xi$. Making box shadow prices equal to zero, the Euler condition on security positions holds if $\lambda^i_\xi = \beta^{t(\xi)+1}\rho^i_\xi(B_{t(\xi)+1} + 1/p_{t(\xi)+1})$, that is, if $(1/p_{t(\xi)+1}) - (1/\beta)(1/p_{t(\xi)}) = -B_{t(\xi)+1}$. The Euler condition on repo positions holds if the repo rate paid at nodes of date t coincides with $p_t B_t$.

We want endowment shocks to have a value 1, in units of account. Hence, $P_{t(\xi)} = 1/p_{t(\xi)}$ and solves the following difference equation:

$$P_{t(\xi)+1} - (1/\beta)P_{t(\xi)} = -B_{t(\xi)+1}$$

[16] Notice that there is no possible confusion between the agent B and the dividends B_t, as the latter are always indexed by the time subscript, while the former not.

Assuming $B_t = k^t$ with $k \in (0,1)$ and positing $P_0 = 1$ the solution becomes

$$P_t = \beta^{-t} + \frac{k}{k-(1/\beta)}(\beta^{-t} - k^t) \tag{6.1}$$

Equilibrium positions: Budget and box constraints at node ξ are, respectively,

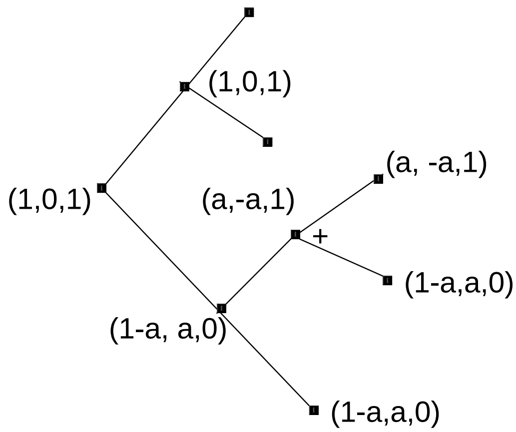
$$p_\xi(x^i_\xi - \omega^i_\xi) + y^i_\xi + hz^i_\xi \leq p_\xi B_{t(\xi)}\varphi^i_{\xi^-} + (1 + p_\xi B_{t(\xi)})hz^i_{\xi^-} \tag{6.2}$$

$$\varphi^i_{\xi^-} + y^i_\xi + \chi_\xi z^i_\xi \geq 0 \tag{6.3}$$

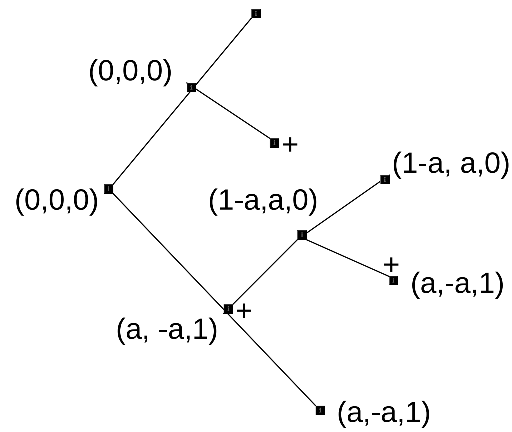
where $\chi_\xi = h$ if $z^i_\xi > 0$ and 1 otherwise. At the initial node ξ_0 we have $(x^A_{\xi_0}, \varphi^A_{\xi_0}, z^A_{\xi_0}) = (\omega^A_{\xi_0}, 1, 0)$ and $(x^B_{\xi_0}, \varphi^B_{\xi_0}, z^B_{\xi_0}) = (\omega^B_{\xi_0}, 0, 0)$. At node ξ_{0d}, agent B uses his endowment shock to purchase the security with funding, that is, $y^B_{\xi_{0d}} + hz^B_{\xi_{0d}} = 1$, which implies $y^B_{\xi_{0d}} = -z^B_{\xi_{0d}} = a$, where $a \equiv 1/(1-h)$. At this node agent A consumes the dividends from the aggregate endowment of the security (and also at node ξ_{0u}). Hence, $x^B_{\xi_{0d}} = \omega^B_{\xi_{0d}} - P_1$, $x^A_{\xi_{0d}} = \omega^A_{\xi_{0d}} + P_1 + B_1$, $\varphi^A_{\xi_{0d}} = 1 - a < 0$ and $z^A_{\xi_{0d}} = a > 1$. At node ξ_{0u}, there are no security or repo trades.

At node ξ_{0du}, agent B consumes the dividends from the aggregate endowment of the security (and also at node ξ_{0dd}). That is, $x^A_{\xi_{0du}} = \omega^A_{\xi_{0du}} - P_2$ and $x^B_{\xi_{0du}} = \omega^B_{\xi_{0du}} + P_2 + B_2$. By (6.2), $y^B_{\xi_{0du}} + hz^B_{\xi_{0du}} + ha = -1$. Agent B's security trade $y^B_{\xi_{0du}} = 1 - 2a$ implies a position $\varphi^B_{\xi_{0du}} = 1 - a$. Then, (6.3) holds for $z^B_{\xi_{0dU}} = a$. By market clearing, we get $\varphi^A_{\xi_{0du}} = a$ and $z^A_{\xi_{0du}} = -a$. At node ξ_{0dd}, there are no security trades and positions are $(\varphi^A_{\xi_{0dd}}, z^A_{\xi_{0dd}}) = (1-a, a)$ and $(\varphi^B_{\xi_{0dd}}, z^B_{\xi_{0dd}}) = (a, -a)$.

At node $\xi_{\xi_{0duu}}$ agent A consumes the dividends from the aggregate endowment of the security (and also at node ξ_{0dud}). Security trades are zero and positions become $(\varphi^A_{\xi_{0duu}}, z^A_{\xi_{0duu}}) = (a, -a)$ and $(\varphi^B_{\xi_{0duu}}, z^B_{\xi_{0duu}}) = (1-a, a)$. At node ξ_{0dud}, security trades are $y^A_{\xi_{0dud}} = 1 - 2a$ and $y^B_{\xi_{0dud}} = 2a - 1$, so $(\varphi^A_{\xi_{0dud}}, z^A_{\xi_{0dud}}) = (1-a, a)$ and $(\varphi^B_{\xi_{0dud}}, z^B_{\xi_{0dud}}) = (a, -a)$. We have determined positions at all types of nodes. See graphs 1 and 2 below.



Graph 1: Agent A's positions $(\varphi^A, z^A, \varphi^A + hz^A)$.



Graph 2: Agent B's positions $(\varphi^B, z^B, \varphi^B + hz^B)$.

To guarantee that the above plans and prices constitute an equilibrium, it just remains to check that the transversality condition (3.3) holds. Using (3.2), this condition can be written as

$$\limsup_T \sum_{\xi:t(\xi)=T} \left(-L^i_{1\xi}(\bar{a}^i) \cdot \bar{a}^i_\xi\right) \leq 0 \tag{6.4}$$

As the condition (3.1) for consumption holds with equality, (6.4) becomes

$$\limsup_T \sum_{\xi:t(\xi)=T} \lambda^i_\xi \left(\varphi^i_\xi + hz^i_\xi\right) \leq 0 \tag{6.5}$$

Now, notice that $\varphi^i_\xi + hz^i_\xi$ coincides with the money position in the example of Páscoa, Petrassi and Torres-Martínez (2011), whereas by (6.1), we have

$$\lambda^i_\xi = \gamma^i_\xi \left(1 + \frac{k}{k-(1/\beta)}(1-(\beta k)^{t(\xi)})\right) \tag{6.6}$$

where γ^i_ξ is the budget multiplier in that example (that is, the multiplier that would make the Euler equation in consumption hold if p_t were β^t), for which $\lim_T \sum_{\xi:t(\xi)=T} \gamma^i_\xi \left(\varphi^i_\xi + hz^i_\xi\right) = 0$. If $k\beta < 1$ then $\lambda^i_\xi < \gamma^i_\xi$ and, therefore, (6.5) holds.

Bubble: In this equilibrium the security has a price bubble. In fact, at each node η the security price 1 is equal to the fundamental value plus the bubble given by $(1/\lambda^i_\eta)\lim_T \sum_{\xi\geq\eta:t(\xi)=T} \lambda^i_\xi$. To evaluate this limit we use (6.6), noticing that $\lim_T(1+\frac{k}{k-(1/\beta)}(1-(\beta k)^{t(\xi)})) > 0$ for $2k < 1/\beta$ and that $\lim_T \sum_{\xi\geq\eta:t(\xi)=T} \gamma^i_\xi = \gamma^i_\eta > 0$ (since fiat money had price 1 and zero fundamental value in the example of

Páscoa, Petrassi and Torres-Martínez (2011)). Notice that the Lagrange deflator process $(\lambda^i_\xi)_\xi$ for which the bubble occurs yields a finite present value of wealth (as already mentioned in Section 3). ♣

Example 2: Let us now consider a different scenario, where agent A is a non-dealer and agent B is a dealer. This corresponds to case II of the analysis above. In order to characterize equilibrium in this framework, we need to do the following modifications with respect to Example 1 above. Now the Euler condition on repo positions holds if both repo rates ($r_{1\xi} - 1$ and $r_{2\xi} - 1$) paid at nodes of date $t(\xi)$ coincides with $p_{t(\xi)} B_{t(\xi)}$. Then, agents A and B' budget constraint are, respectively,

$$p_\xi(x^A_\xi - \omega^A_\xi) + y^A_\xi + \theta^A_\xi - h\psi^A_\xi \leq p_\xi B_{t(\xi)} \varphi^A_{\xi^-} + (1 + p_\xi B_{t(\xi)})(\theta^A_{\xi^-} - h\psi^A_{\xi^-}) \quad (6.7)$$

$$p_\xi(x^B_\xi - \omega^B_\xi) + y^B_\xi + h\theta^B_\xi - \psi^B_\xi \leq p_\xi B_{t(\xi)} \varphi^B_{\xi^-} + (1 + p_\xi B_{t(\xi)})(h\theta^B_{\xi^-} - \psi^B_{\xi^-}) \quad (6.8)$$

Agents A and B' box constraints are, respectively,

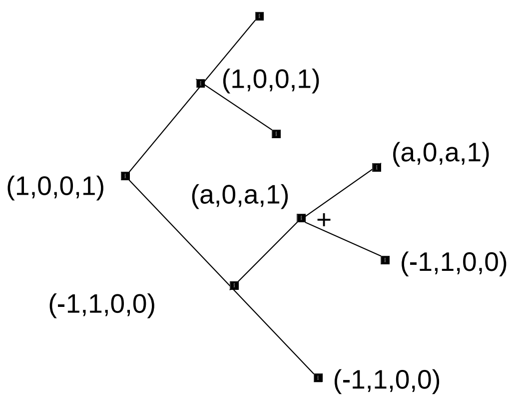
$$\varphi^A_\xi + \theta^A_\xi - \psi^A_\xi \geq 0 \quad (6.9)$$

$$\varphi^B_\xi + \theta^B_\xi - \psi^B_\xi \geq 0 \quad (6.10)$$

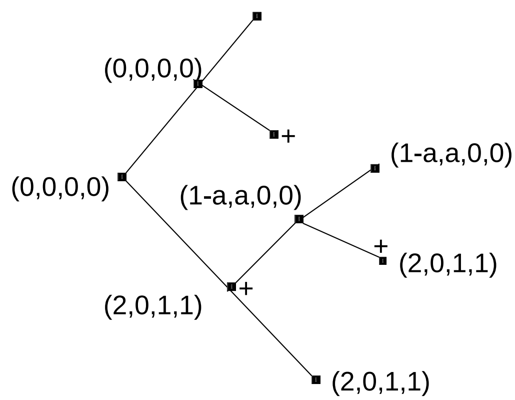
The equilibrium positions are now the following. At the initial node ξ_0 there are no securities nor repo trades, so $(x^A_{\xi_0}, \varphi^A_{\xi_0}, \theta^A_{\xi_0}, \psi^A_{\xi_0}) = (\omega^A_{\xi_0}, 1, 0, 0)$ and $(x^B_{\xi_0}, \varphi^B_{\xi_0}, \theta^B_{\xi_0}, \psi^B_{\xi_0}) = (\omega^B_{\xi_0}, 0, 0, 0)$. At node ξ_{0d}, agent B has a positive shock and uses this extra endowment to lend cash through repo, so $\psi^B_{\xi_{0d}} > 0$ and $\theta^B_{\xi_{0d}} = 0$. Agent B's budget and box constraints require that $y^B_{\xi_{0d}} - \psi^B_{\xi_{0d}} = 1$ and $y^B_{\xi_{0d}} - \psi^B_{\xi_{0d}} \geq 0$, respectively. We take $\psi^B_{\xi_{0d}} = 1$ and $\varphi^B_{\xi_{0d}} = 2$. Now, agent A's box constraint holds with equality if we take $\theta^A_{\xi_{0d}} = 1$ and $\varphi^A_{\xi_{0d}} = -1$. Therefore, $(\varphi^A_{\xi_{0d}}, \theta^A_{\xi_{0d}}, \psi^A_{\xi_{0d}}) = (-1, 1, 0)$ and $(\varphi^B_{\xi_{0d}}, \theta^B_{\xi_{0d}}, \psi^B_{\xi_{0d}}) = (2, 0, 1)$. At node ξ_{0u}, there are no securities nor repo trades. Thus, each agent consumes his endowments and the portfolio positions remain, $(\varphi^A_{\xi_{0u}}, \theta^A_{\xi_{0u}}, \psi^A_{\xi_{0u}}) = (1, 0, 0)$ and $(\varphi^B_{\xi_{0u}}, \theta^B_{\xi_{0u}}, \psi^B_{\xi_{0u}}) = (0, 0, 0)$.

At node ξ_{0du}, agent B's budget and box constraints require $\varphi^B_{\xi_{0du}} + h\theta^B_{\xi_{0du}} = 0$ and $\varphi^B_{\xi_{0du}} + \theta^B_{\xi_{0du}} \geq 0$, respectively. We make $\varphi^B_{\xi_{0du}} = 1 - a$ and $\theta^B_{\xi_{0du}} = a$. Thus, for agent A it must be that $\varphi^A_{\xi_{0du}} = a$ and $\psi^A_{\xi_{0du}} = a$. Therefore, $(\varphi^A_{\xi_{0du}}, \theta^A_{\xi_{0du}}, \psi^A_{\xi_{0du}}) = (a, 0, a)$ and $(\varphi^B_{\xi_{0du}}, \theta^B_{\xi_{0du}}, \psi^B_{\xi_{0du}}) = (1 - a, a, 0)$. At node ξ_{0dd}, there are no security nor repo trades, and therefore $(\varphi^A_{\xi_{0dd}}, \theta^A_{\xi_{0dd}}, \psi^A_{\xi_{0dd}}) = (-1, 1, 0)$ and $(\varphi^B_{\xi_{0dd}}, \theta^B_{\xi_{0dd}}, \psi^B_{\xi_{0dd}}) = (2, 0, 1)$.

At node $\xi_{\xi_{0duu}}$, portfolio positions are the same as in node $\xi_{\xi_{0du}}$, whereas at $\xi_{\xi_{0dud}}$ positions are the same as in node $\xi_{\xi_{0d}}$. We have determined positions at all types of nodes. See graphs 3 and 4 below.



Graph 3: Agent A's positions $(\varphi^A, \theta^A, \psi^A, \varphi^A + \theta^A - h\psi^A)$.



Graph 4: Agent B's positions $(\varphi^B, \theta^B, \psi^B, \varphi^B + \theta^B - h\psi^B)$.

It remains to show that the transversality condition (3.3) holds for both agents. For A and B, these conditions are respectively

$$\limsup_T \sum_{\xi : t(\xi) = T} \lambda^i_\xi \left(\varphi^A_\xi + \theta^A_\xi - h\psi^A_\xi \right) \leq 0 \tag{6.9}$$

and

$$\limsup_T \sum_{\xi : t(\xi) = T} \lambda^i_\xi \left(\varphi^B_\xi + h\theta^A_\xi - \psi^B_\xi \right) \leq 0 \tag{6.10}$$

We now notice that $\varphi^A_\xi + \theta^A_\xi - h\psi^A_\xi$ and $\varphi^B_\xi + h\theta^A_\xi - \psi^B_\xi$ coincide with the term $\varphi^i_\xi + hz^i_\xi$ of condition (6.5) in Example 1. ♣

In the next section we discuss whether there are other non-arbitrage deflators for which the above security prices are free of bubbles.

Remark on uniform impatience: The well known result that bubbles in positive net supply assets are absent under uniform impatience for deflators with finite present value of wealth (Santos and Woodford (Theorem 3.3)) extends to securities that can only be shorted by borrowing them first.

Definition 2: We say that the economy satisfies *uniform impatience* if there exist $\pi \in (0,1)$ and Δ_ξ for each $\xi \in D$ such that, for all $x \in \mathbb{R}^{LD}_+$ with $x \leq \Omega$, we have $U^i(\tilde{x}(\xi, \pi')) > U^i(x)$, $\forall i \in \mathbf{I}$, $\xi \in D$, where $\tilde{x}_\eta(\xi, \pi') = x_\eta$ for $\eta \in D \backslash D(\xi)$,

$\tilde{x}_\xi(\xi, \pi') = x_\xi + \Delta_\xi$ and $\tilde{x}_\eta(\xi, \pi') = \pi' x_\eta$ for $\eta > \xi$ with $\pi' \in [\pi, 1)$. Moreover, $\exists k > 0 : \omega^i_\xi \geq k\Delta_\xi > 0$, $\forall i \in \mathbf{I}$, $\xi \in D$.

Let us show that in *case I* uniform impatience implies absence of price bubbles for a security in positive net supply and a deflator with finite present value of wealth.

Denote by (p, q, r, x, y, z) an equilibrium and consider the following change in consumption and financial plans: $x^i_\xi \mapsto x^i_\xi + \Delta_\xi$, $x^i_\eta \mapsto \pi x^i_\eta$ for $\eta > \xi$, $(\varphi^i_\eta, z^i_\eta) \mapsto \pi(\varphi^i_\eta, z^i_\eta)$ for $\eta \geq \xi$. Under uniform impatiance an appropiate choice of (Δ_ξ, π) requires $(1 - \pi)q_\xi(\varphi^i_\xi + h_\xi z^i_\xi) < p_\xi \Delta_\xi$, so that optimality is not contradicted. Notice that in case I (see proof of Proposition 3) we have $\varphi^i_\xi + h_\xi z^i_\xi \geq 0$. It follows that $0 \leq (q_\xi/p_\xi\Delta_\xi)(\varphi^i_\xi + h_\xi z^i_\xi) < 1/(1-\pi)$. Adding across consumers we see that $q_\xi/p_\xi\Delta_\xi$ is uniformly bounded. If γ is such that $\sum_{t=0}^\infty \sum_{\xi:t(\xi)=t} \gamma_\xi p_\xi \omega^i_\xi < \infty$, using again Definition 2, we get $\lim_{t\to\infty} \sum_{\xi:t(\xi)=t} \gamma_\xi q_\xi = 0$ as claimed.

In *case II* we take the equilibrium plans of *non-dealers* and modify them as above. Now, $(\varphi^i_\eta, \theta^i_\eta, \psi^i_\eta) \mapsto \pi(\varphi^i_\eta, \theta^i_\eta, \psi^i_\eta)$,for $\eta \geq \xi$. Again, we require $(1 - \pi)q_\xi(\varphi^i_\xi + \theta^i_\xi - h_\xi\psi^i_\xi) < p_\xi\Delta_\xi$, where $\varphi^i_\xi + \theta^i_\xi - h_\xi\psi^i_\xi \geq 0$ by the box constraint. So, it follows that $(q_\xi/p_\xi\omega^m_\xi)\sum_{i\in\mathbf{ND}}(\varphi^i_\xi + \theta^i_\xi - h_\xi\psi^i_\xi) < k^{-1}(\sharp\mathbf{ND})/(1-\pi)$, for any $m \in \mathbf{ND}$.

Now, by (A3) we know that $(q_\xi/\sum_l p_{l\xi})\sum_{i\in\mathbf{D}} \theta^i_\xi$ is uniformly bounded and, therefore, $(q_\xi/\sum_l p_\xi)\sum_{i\in\mathbf{D}} \varphi^{i-}_\xi$ is also uniformly bounded (by the box constraint). It follows that $(q_\xi/\sum_l p_\xi)\sum_{i\in\mathbf{D}}(\varphi^i_\xi - \psi^i_\xi + h_\xi\theta^i_\xi)$ is uniformly bounded.[17] Putting the two results together we have that $(q_\xi/p_\xi\omega^m_\xi)(\sum_{i\in\mathbf{I}} \varphi^i_\xi + \sum_{i\in\mathbf{ND}} \theta^i_\xi - \sum_{i\in\mathbf{D}} \psi^i_\xi - h_\xi(\sum_{i\in\mathbf{ND}} \psi^i_\xi - \sum_{i\in\mathbf{D}} \theta^i_\xi))$ is uniformly bounded (as $\sum_l p_{l\xi}/p_\xi\omega^m_\xi \leq 1/\inf_\xi \omega^m_\xi$). Hence, $(q_\xi/p_\xi\omega^m_\xi)_\xi \in l^\infty$ and we get $\lim_{t\to\infty} \sum_{\xi:t(\xi)=t} \gamma_\xi q_\xi = 0$ as claimed for γ with $\sum_{t=0}^\infty \sum_{\xi:t(\xi)=t} \gamma_\xi p_\xi \omega^m_\xi < \infty$.

7 Specialness and bubbles

As seen in equation (15), the security price at a node ξ may exceed the series of discounted dividends for two reasons. First, there may be positive shadow prices of box constraints at this or future nodes. Second, the security may have a price at infinity. The former raises the fundamental value above what that discounted

[17] Since the uniform bounds from below and from above follow, respectively, from $\varphi^i_\xi - \psi^i_\xi \geq -\theta^i_\xi$ and $\sum_{i\in\mathbf{D}}(\varphi^i_\xi - \psi^i_\xi) = \sum_{i\in\mathbf{ND}}(\varphi^i_\xi - \theta^i_\xi) < \sum_{i\in\mathbf{ND}}(\varphi^{i+}_\xi - \theta^i_\xi) = \sum_{i\in\mathbf{D}}(\varphi^{i-}_\xi - \psi^i_\xi)$.

dividends series is. The latter creates a bubble.

A positive shadow price of the box constraint at a node ξ implies that the security is on special. This means that the respective repo rate is below the general collateral rate, which is the prevailing repo market interest rate if the borrower of funds can choose the bond to pledge. GC is generally below uncollateralized borrowing, taken here, for simplicity, to be the interest rate (ι_ξ) on a risk free one-period bond. In fact, adding this bond to the model and adapting the argument in Bottazzi, Luque and Páscoa (2011) to case I, say, the Euler conditions on repo positions of security j and on trade of that risk-free bond are, respectively,

$$1 = \frac{\mu^i_{j\xi}\chi^i_{j\xi}}{q_{j\xi}h_{j\xi}\lambda^i_\xi} + (1+\rho_\xi)\sum_{\eta\in\xi^+}\frac{\lambda^i_\eta}{\lambda^i_\xi} \tag{17}$$

$$1 = (1+\iota_\xi)\sum_{\eta\in\xi^+}\frac{\lambda^i_\eta}{\lambda^i_\xi} \tag{18}$$

Condition (18) holds in equality form as we are comparing with a risk free bond that can be shorted and is itself not on special (see Duffie (1996), p. 494, where such comparison is also done). Hence, $\rho_\xi < \iota_\xi$ if and only if $\mu^i_{j\xi} > 0$. So specialness makes the security price to be above its series of discounted dividends. Can the bubble occur on top or just instead of this overpricing? The next result shows that it can but the occurrence of a bubble requires specialness to fade away as time goes to infinity.

Proposition 5: *If security j has a bubble, at some node ξ, then*

$$\lim_{T\to\infty}\max_{\eta>\xi:t(\eta)=T}(\iota_{j\eta}-\rho_{j\eta}) = 0.$$

The proof of Proposition 5 is in the Appendix.[18]

Remark to Examples 1 and 2: Are there other deflators for which the equilibrium security prices found in Examples 1 and 2 are free of bubbles?[19]

[18] Observe that $\lim_{T\to\infty}\sum_{\eta:t(\eta)=T}\mu^i_{j\eta} \to 0$ does not, by itself, imply $\lim_T \max_{\eta>\xi:t(\eta)=T}(\iota_{j\eta} - \rho_{j\eta}) = 0$, as we just know that $\lim_T \max_{\eta>\xi:t(\eta)=T}\frac{1+\rho_\eta}{1+\iota_\eta} \geq 1 - \lim_{T\to\infty}\frac{\max_{\eta>\xi:t(\eta)=T}\mu^i_{j\xi}}{\underline{\mathrm{h}}_T \min_{\eta>\xi:t(\eta)=T}\lambda^i_\xi q_{j\xi}}$, where $\min_{\eta>\xi:t(\eta)=T}\lambda^i_\xi q_{j\xi}$ may tend to zero.

[19] We thank Manuel Santos for having risen this issue.

Santos and Woodford (1997, Theorem 3.1) showed that when the supremum, over all non-arbitrage deflators, of the present value of wealth is finite, there is always a deflator for which bubbles are absent. Such non-arbitrage deflators were defined by requiring *just* the Euler condition on asset positions to hold *with equality*. For the Kuhn-Tucker deflator that we use in the examples, all Euler conditions hold (namely, on consumption).

Our examples were based on an example (in Páscoa, Petrassi, Torres-Martínez (2011)) of an economy with fiat money that fits in Santos and Woodford (1997) set-up, but money has a positive price due to a bubble. Clearly, such equilibrium price cannot be obtained free of bubbles for another deflator with non-arbitrage conditions holding with equality (as the price would be zero, since money has no dividends). Hence, by Theorem 3.1 in Santos and Woodford (1997), the supremum over all such deflators of the present value of wealth had to be infinite in that monetary example. Nevertheless, the positive price of money can be shown to be recovered free of bubbles for a deflator with non-arbitrage conditions holding with strict inequality at nodes where the portfolio constraint (a no-short-sales restriction on money) was binding.

Similarly, in the examples in this paper, we can find a non-arbitrage deflator for which the equilibrium security price is free of bubbles, but this deflator makes non-arbitrage conditions hold with strict inequality at nodes where the box constraint is binding. In this way, we keep the same security prices, but the bubble is replaced by an extra term in the fundamental value consisting of the series of box shadow prices.

Let us be more precise and start by defining a non-arbitrage deflator.

Definition 3: *In case I, we say that for prices (q, r) there are linearized arbitrage opportunities at node ξ if $\exists\ (\varphi_\xi, z_\xi) : \varphi_\xi + \chi_\xi z_\xi \geq 0$ and $A(\varphi_\xi, z_\xi) > 0$, where $\chi_\xi = H_\xi$ if $z_\xi > 0$, $\chi_\xi = 1$ if $z_\xi < 0$ and $\chi_\xi \in [H_\xi, 1]$, and*

$$A = \begin{bmatrix} -q_\xi & -q_\xi h_\xi \\ (q_\eta + p_\eta B_\eta)_{\eta \in \xi^+} & (h_\xi r_\xi q_\xi)_{\eta \in \xi^+} \end{bmatrix}$$

Lemma 2: *In case I, there exist no arbitrage opportunities if and only if*

$\exists \alpha \gg 0, \nu \geq 0$ *such that*

$$
\begin{aligned}
\alpha_\xi q_\xi &= \sum_{\eta \in \xi^+} \alpha_\eta (q_\eta + p_\eta B_\eta) + \nu_\xi \\
\alpha_\xi q_\xi h_\xi &= \sum_{\eta \in \xi^+} \alpha_\eta r_\xi h_\xi q_\xi + \chi_\xi \nu_\xi
\end{aligned}
$$

This is proven by the theorem of separation of convex cones (see Araujo, Fajardo and Páscoa (2002, Theorem 1).

Let us call (α, υ) a pair of non-arbitrage deflator and shadow values.

Proposition 6: *In Example 1, the same security prices q are obtained if the Kuhn-Tucker deflator λ is replaced by the non-arbitrage deflator $\alpha_\xi^i = \left(\beta^{t(\xi)}/p_{t(\xi)}\right) \sum_{\eta \geq \xi} \mu_\eta^i$ together with the shadow values process $\nu_\xi = \mu_\xi^i \beta^t / p_t$, where $\mu_\xi^i = \rho_\xi^i 2^{-t(\xi)}$ if the box constraint is binding at node ξ and $z_\xi^i > 0$, and $\mu_\xi^i = 0$ otherwise.*

Definition 4: *In case II, we say that for prices (q, r) there are arbitrage opportunities*

- for a dealer at node ξ if $\exists$ $(\varphi_\xi, \theta_\xi, \psi_\xi) : \theta_\xi \geq 0$, $\psi_\xi \geq 0$, $\varphi_\xi + \theta_\xi - \psi_\xi \geq 0$ and $A^d(\varphi_\xi, \theta_\xi, \psi_\xi) > 0$, where

$$
A^d = \begin{bmatrix} -q_\xi & -h_\xi q_\xi & -q_\xi \\ (q_\eta + p_\eta B_\eta)_{\eta \in \xi^+} & (h_\xi r_{1\xi} q_\xi)_{\eta \in \xi^+} & (r_{2\xi} q_\xi)_{\eta \in \xi^+} \end{bmatrix}
$$

- for a non-dealer at node ξ if $\exists$ $(\varphi_\xi, \theta_\xi, \psi_\xi) : \theta_\xi \geq 0$, $\psi_\xi \geq 0$, $\varphi_\xi + \theta_\xi - \psi_\xi \geq 0$ and $A^{nd}(\varphi_\xi, \theta_\xi, \psi_\xi) > 0$, where

$$
A^{nd} = \begin{bmatrix} -q_\xi & -q_\xi & -h_\xi q_\xi \\ (q_\eta + p_\eta B_\eta)_{\eta \in \xi^+} & (r_{1\xi} q_\xi)_{\eta \in \xi^+} & (h_\xi r_{2\xi} q_\xi)_{\eta \in \xi^+} \end{bmatrix}
$$

Lemma 3: *In case II, there exist no arbitrage opportunities if and only if $\exists \alpha \gg 0, \nu \geq 0$ such that, if the agent is a dealer, we have*

$$
\begin{aligned}
\alpha_\xi q_\xi &= \sum_{\eta \in \xi^+} \alpha_\eta (q_\eta + p_\eta B_\eta) + \nu_\xi && (19) \\
\alpha_\xi q_\xi h_\xi &\geq \sum_{\eta \in \xi^+} \alpha_\eta r_{1\xi} h_\xi q_\xi + \nu_\xi && (20) \\
\alpha_\xi q_\xi &\geq \sum_{\eta \in \xi^+} \alpha_\eta r_{2\xi} q_\xi + \nu_\xi && (21)
\end{aligned}
$$

If the agent is a non-dealer instead, we replace ν_ξ by $h_\xi \nu_\xi$ in (20) and ν_ξ by ν_ξ / h_ξ in (21).

An analogous proposition to Proposition 6 for case II holds with $\mu^i_\xi = \rho^i_\xi 2^{-t(\xi)}$ if the box constraint is binding at node ξ and $z^i_\xi > 0$, and $\mu^i_\xi = 0$ otherwise.

Finally, notice that the supremum of the present value of aggregate endowments, over all deflators for which non-arbitrage conditions hold with equality, is still infinite as in the example in Páscoa, Petrassi and Torres-Martínez (2011). In fact, the way that supremum is computed (indicated by Proposition 2.2 in Santos and Woodford (1997)) in that example shows that the supremum for the economy with fiat money is the same as the supremum in the economy with repo markets. This suggests that when we look for a deflator for which there is no bubble, we should find it within the class where non-arbitrage conditions may hold with strict inequalities at some nodes.

8 Concluding remarks

Once we explicitly take into account the way securities are actually shorted, by borrowing them first in repo markets, we see that mechanisms that bound leverage and ensure equilibrium in finite horizon will also prevent infinite lived agents from doing Ponzi schemes. This result dispenses any uniform impatience assumptions. In this security and repo context, we see reappearing the main insight in Santos and Woodford (1996): bubbles in positive net supply securities (for deflators with finite present value of wealth) cannot occur when markets are complete but may occur in incomplete markets when consumers are not uniformly impatient. However, that room for bubbles seemed to be quite narrow before, as, in the absence of uniform impatience, short sales apparently had to be ruled out (as in the examples by Santos and Woodford (1996) or Páscoa, Petrassi and Torres-Martínez (2011)).

Now, if shorting and security borrowing are properly coupled, Ponzi schemes are not always ruled out (as our example shows) but are absent when there is limited leverage. So, bubbles may occur in incomplete markets, when uniform impatience fails but leverage is adequately bounded. We presented two ways that limit the re-hypothecation of the security and the resulting leverage. One is the increasing practise (after Lehman's banruptcy) of not re-using (shorting or lending) the haircut collected when borrowing a security. The other is the current arrangement that limits, by regulation, the positions of dealers, whom by collecting but not posting haircut would have an incentive to borrow and lend, at the same time, large amounts of securities. Such non-convexity in dealers' attitudes contrasts with their counterparties' preference for convex combinations, as haircut is posted but not collected. Counterparties (hedge funds, mutual funds, retail securities brokers, private banks and insurance companies) do not need to have positions bounded.

There are many issues that would deserve further work. As usual in the literature, the assets' positive net supply results from initial holdings at the first date. Issuance at other nodes of the event-tree is not being considered. An interesting step would be to model issuance and discuss its implications (both in the equity and the debt forms). Clearly, the issuance chosen should not be bounded by simple quantitative constraints (of the type that bounded borrowing in the previous literature) but one should take into account that a large issuance may decrease the security price (raise the interest paid on debt). Price taking might be questionable in that context. Repo fails or the counterparties' default in repurchasing securities were also not addressed, but there may be interesting substitution effects between not honoring repo agreements and running a Ponzi scheme. Finally, more applied work could try to look at the relative importance of bubbles and specialness as two forms of overpricing securities, both shown by us to occur only in incomplete markets.

9 Appendix

Proof of Proposition 1: For each agent i and for each $T \in \mathbb{N}$, we define , *an optimization problem with finite horizon* T by truncating the utility functions in the following way $U^{iT}(x) = \sum_{\xi \in D^T(\xi_0)} u^i_\xi(x_\xi)$ and modifying the budget and box constraints at ξ with $t(\xi) \geq T-1$ in the following way. For $t > T$ no commodity, security or repo trades can be done. At $t = T$ commodities can be traded, securities pay dividends but are no longer traded. At $T-1$ repo trades cannot be done (since at the following date securities have no value) and securities are traded under a plain no-short-sales restriction.

That is, for ξ with $t(\xi) = T-1$ we require $g^{iT}(a_\xi, a_{\xi^-}; p, q, r) \equiv p_\xi(x_\xi - \omega^i_\xi) + q_\xi(\varphi_\xi - \varphi_{\xi^-}) - p_\xi B_\xi \varphi_{\xi^-} - q_{\xi^-} r_\xi h_{\xi^-} z_{\xi^-} \leq 0$ together with $f^{iT}(a_\xi, a_{\xi^-}) \equiv \varphi_\xi \geq 0$. For ξ such that $t(\xi) = T$ we require $g^{iT}(a_\xi, a_{\xi^-}; p, q, r) \equiv p_\xi(x_\xi - \omega^i_\xi) - q_\xi \varphi_{\xi^-} -$

$p_\xi B_\xi \varphi_{\xi^-} \leq 0.$

The truncated Lagrangean is defined at each node ξ by

$$L_\xi^{iT}(a_\xi, a_{\xi^-}; \lambda_\xi, \mu_\xi, p, q, r) \equiv u_\xi^i(x_\xi) - \lambda_\xi g_\xi^{iT}(a_\xi, a_{\xi^-}; p, q, r) + \mu_\xi f_\xi^{iT}(a_\xi)$$

Now, the saddle point property holds for some multipliers $(\lambda_\xi^{iT}, \mu_{j\xi}^{iT})_{j\xi}$ at a solution a^{iT} to this truncated problem[20], that is, for any plan $(a_\xi)_{\xi \in D^T(\xi_0)}$ satisfying relevant sign constraints we have

$$\sum_{\xi: t(\xi) \leq T} L_\xi^{iT}(a_\xi, a_{\xi^-}; \lambda_\xi^{iT}, \mu_\xi^{iT}, p, q, r) \leq U^{iT}\left(\bar{x}^{iT}\right) \tag{22}$$

$$\lambda_\xi^{iT} g_\xi^{iT}(\bar{a}_\xi, \bar{a}_{\xi^-}; p, q, r) = 0, \quad \mu_\xi^{iT} f_\xi^{iT}(\bar{a}_\xi) = 0. \tag{23}$$

Notice that $U^{iT}\left(\bar{x}^{iT}\right) \leq U^i(\bar{x}^i)$ (as the truncated problem has additional constraints), so

$$\sum_{\xi: t(\xi) \leq T} L_\xi^{iT}(a_\xi, a_{\xi^-}; \lambda_\xi^{iT}, \mu_\xi^{iT}, p, q, r) \leq U^i(\bar{x}^i) \tag{24}$$

(i) To derive *Euler conditions*, as $\mu_\xi^{iT} f_\xi^{iT}(\bar{\varphi}_\xi^i, \bar{z}_\xi^i) = 0$, we can work with the following inequality

$$\sum_{\xi: t(\xi) \leq T} L_\xi^{iT}(a_\xi, a_{\xi^-}; \lambda_\xi^{iT}, \mu_\xi^{iT}, p, q, r) \leq U^i(\bar{x}^i) + \sum_{\xi: t(\xi) \leq T} \mu_\xi^{iT} f_\xi^{iT}(\bar{\varphi}_\xi^i, \bar{z}_\xi^i) \tag{25}$$

The next claim is almost the desired Euler conditions as it says that $\lambda_\xi^{iT} \mathcal{D}_1 g_\xi^{iT} + \sum_{\eta \in \xi^+} \lambda_\eta^{iT} \mathcal{D}_2 g_\eta^{iT}$ is getting closer, as $T \to \infty$, to being a super-gradient of $u_\xi^i + \mu_\xi f_\xi^{iT}$ (with u_ξ^i restricted to the positive orthant).

Claim 1: *For each ξ with $t(\xi) \leq T-1$ and for any action for that node $b = (\tilde{x}_\xi, \tilde{\varphi}_\xi, \tilde{z}_\xi)$ with $\tilde{x}_\xi \geq 0$ we have*

$$u_\xi^i(\tilde{x}_\xi) - u_\xi^i(\bar{x}_\xi^i) + \mu_\xi(f_\xi^{iT}(\tilde{\varphi}_\xi, \tilde{z}_\xi) - f_\xi^{iT}(\bar{\varphi}_\xi^i, \bar{z}_\xi^i)) \leq (\lambda_\xi^{iT} \mathcal{D}_1 g_\xi^{iT}(\bar{a}_\xi^i, \bar{a}_{\xi^-}^i; p, q, r) + \sum_{\eta \in \xi^+} \lambda_\eta^{iT} \mathcal{D}_2 g_\eta^{iT}(\bar{a}_\eta^i, \bar{a}_{\eta^-}^i; p, q, r))(b - \bar{a}_\xi^i) + \sum_{\eta \in D \setminus D^T} u_\eta^i(\bar{x}_\eta^i) \tag{26}$$

[20] This follows by the generalized Slater constraint qualification (Uzawa (1958)): by making $x_{\xi_0} = \omega_{\xi_0}^i$, $\varphi_\xi = e_\xi^i$ and for $\xi \neq \xi_0$, $x_\xi = \omega_\xi^i + B_\xi e^i$, so that budget constraints hold with equality and box constraints with strict inequality.

Proof of Claim 1: This follows by using (25) making, for any node η, $a_\eta = \bar{a}^i_\eta(1-\mathbf{1}_\xi(\eta)) + b\mathbf{1}_\xi(\eta)$ where $\mathbf{1}_\xi(\eta) = 1$ if $\eta = \xi$ and equal to 0 otherwise, for ξ with $t(\xi) \leq T-1$. ■

Now, we let $T \to \infty$. The sequence $(\lambda^{iT}_\xi, (\mu^{iT}_{j\xi})_j)_T$ is bounded for each node ξ by Lemma 4 below. So, we can find a cluster point $(\lambda^i_\xi, (\mu^i_{j\xi})_j)$ for the countable product topology. Denote by δ the indicator function of $\mathbb{R}^L_+$ defined by $\delta(c) = 0$ for $c \in \mathbb{R}^L_+$ and $\delta(c) = -\infty$ otherwise. Let $u^i_\xi(.) + \delta(.) \equiv \hat{u}^i_\xi(.)$. Taking the limit in (26) we get that $\lambda^i_\xi \mathcal{D}_1 g^i_\xi(\bar{a}^i_\xi, \bar{a}^i_{\xi^-}; p, q, r) + \sum_{\eta\in\xi^+} \lambda^i_\eta \mathcal{D}_2 g^i_\eta(\bar{a}^i_\eta, \bar{a}^i_{\eta^-}; p, q, r)$ is a supergradient v'_ξ at point $\bar{a}^i_\xi$ of the function $v_\xi : (\tilde{x}_\xi, \tilde{\varphi}_\xi, \tilde{z}_\xi) \mapsto \hat{u}^i_\xi(\tilde{x}^i_\xi) + \mu_\xi f^i_\xi(\tilde{\varphi}_\xi, \tilde{z}_\xi)$, which embodies the restriction of u^i_ξ to the positive orthant .

Now, k is a super-gradient of δ at c iff $k(c'-c) \geq 0$ for any $c' \geq 0$. So $kc = 0$ (by picking $c' = 0$ and $c' = 2c$). Applying Theorem 23.8 in Rockafellar (1970) we write v'_ξ as a sum of a supergradient of the unrestricted function $u^i_\xi(.) + \mu_\xi f^i_\xi(.)$ and a supergradient k of δ. We get (10) from the non-negativity of k and (11) from $k\bar{a}^i_\xi = 0$.

Observe that the equality in (10) actually holds for the coordinates of variables not subject to sign constraints.

(ii) Let us now prove that the *transversality condition* (12) holds. We use the inequality (24) and make $a_\xi = \bar{a}^i_\xi$ if $t(\xi) \leq t-1$ and $a_\xi = 0$ otherwise. Notice that $g^i_\xi(0, \bar{a}_{\xi^-}; p, q, r) = -p_\xi\omega^i_\xi + \mathcal{D}_2 g^i_\xi(p, q, r)\cdot\bar{a}^i_{\xi^-}$, for ξ with $t(\xi) = t$, and $g^i_\xi(0, 0; p, q, r) = -p_\xi\omega^i_\xi$, for ξ with $t(\xi) \geq t+1$, as $u^i(\xi, 0) = 0$. Then, we obtain

$$-\sum_{\xi:t(\xi)=t} \lambda^{iT}_\xi \cdot \mathcal{D}_2 g^i_\xi(p, q, r)\cdot\bar{a}^i_{\xi^-} + \sum_{\xi:T\geq t(\xi)>t} \lambda^{iT}_\xi p_\xi\omega^i_\xi \leq \sum_{\xi:t(\xi)\geq t} u^i_\xi(\bar{x}^i_\xi)$$

Now, by (A1)(ii) the series of utilities converges for feasible plans (that is, $\sum_{\xi:t(\xi)\geq t} u^i_\xi(\bar{x}^i_\xi) \to 0$ as $t \to \infty$). Then, using $L^i_{2\xi}(a_\xi, a_{\xi^-}) = -\lambda^i_\xi \cdot \mathcal{D}_2 g^i_\xi(p, q, r)$, we obtain (12). ■

Proof of Proposition 2: Consider any plan a^i satisfying the budget, box and sign constraints. Let x^i its respective consumption plan. Denote by U^{iT} the truncation of U^i to the finite horizon T. Now, we know that

$$U^{iT}(x) - U^{iT}(\bar{x}^i) \leq \sum_{\xi:t(\xi)\leq T} (L^i_\xi(a^i) - L^i_\xi(\bar{a}^i)). \tag{27}$$

Notice that by the superdifferential property,

$$\sum_{\xi:t(\xi)\le T}(L^i_\xi(a^i)-L^i_\xi(\bar a^i))\le \sum_{\xi:t(\xi)\le T}(L^i_{1\xi}(\bar a^i),L^i_{2\xi}(\bar a^i))((a^i_\xi,a^i_{\xi^-})-(\bar a^i_\xi,\bar a^i_{\xi^-}))=$$
$$=\sum_{\xi:t(\xi)<T}(L^i_{1\xi}(\bar a^i)+\sum_{\eta\in\xi^+}L^i_{2\eta}(\bar a^i))a^i_\xi+\sum_{\xi:t(\xi)=T}L^i_{1\xi}(\bar a^i)a^i_\xi-$$
$$-\sum_{\xi:t(\xi)<T}(L^i_{1\xi}(\bar a^i)+\sum_{\eta\in\xi^+}L^i_{2\eta}(\bar a^i))\bar a^i_\xi-\sum_{\xi:t(\xi)=T}L^i_{1\xi}(\bar a^i)\bar a^i_\xi$$

Taking $T\to\infty$ and using Euler and transversality conditions (10), (11) and (12), we get

$$\limsup_{T\to\infty}\sum_{\xi:t(\xi)\le T}(L^i_\xi(a)-L^i_\xi(\bar a^i))\le \limsup_{T\to\infty}\sum_{\xi:t(\xi)=T}L^i_{1\xi}(\bar a^i)a_\xi \tag{28}$$

and, therefore,

$$\limsup_{T\to\infty}(U^{iT}(x)-U^{iT}(\bar x^i))\le \limsup_{T\to\infty}\sum_{\xi:t(\xi)=T}L^i_{1\xi}(\bar a^i)a_\xi \tag{29}$$

■

Proof of Lemma 1: We take inequality (24) and make $a_\eta=(0,0,0)$ for every node η. Then,

$$\sum_{\eta:t(\eta)\le T}\lambda^i_\eta p_\eta\omega^i_\eta+\lambda^i_{\xi_0}q_{\xi_0}e^i\le U^i(\bar x^i)$$

Making $T\to\infty$ we see that $\sum_{\eta:t(\eta)\le T}\lambda^i_\eta p_\eta\omega^i_\eta$ converges. ■

Proof of Proposition 3: The Euler condition on $x_\xi\ge 0$ (which holds at this cluster point of finite horizon equilibria) is $\frac{\partial L^i_\xi}{\partial x_\xi}=\mathcal{D}_{x_\xi}u^i_\xi(\xi,x^i_\xi)-\lambda^i_\xi p_\xi\le 0$ for all $\xi\in D$. Therefore, (13) can be rewritten as shown next.

In case I: Let $\mathcal{A}^i_\xi\equiv-\lambda^i_\xi q_\xi(\varphi_\xi+h_\xi z_\xi)+\mu^i_\xi(\varphi_\xi+H_\xi z^+_\xi-z^-_\xi)$. Then (13) reduces to requiring

$$\limsup_{T\to\infty}\sum_{\xi:t(\xi)=T}\mathcal{A}^i_\xi\le 0$$

to hold for any (φ,z) satisfying budget and box constraints. Notice that $\varphi_\xi+H_\xi z^+_\xi-z^-_\xi\ge 0$ implies $\varphi_\xi+h_\xi z_\xi\ge 0$, as $H_\xi\le h_\xi<1$. So, if $\mu^i_\xi=0$, then $\mathcal{A}^i_\xi\le 0$, $\forall\xi\in D$. But if $\mu^i_\xi>0$, then $\mathcal{A}^i_\xi=(-\lambda^i_\xi q_\xi+\mu^i_\xi)\varphi_\xi+(-\lambda^i_\xi q_\xi h_\xi+\mu^i_\xi\chi_\xi)z_\xi$, $\forall\xi\in D$, where $\chi_\xi=H_\xi$ if $z_\xi>0$ and 1 otherwise.

Now, $\mathcal{A}^i_\xi \leq (-\lambda^i_\xi q_\xi + \mu^i_\xi)(\varphi_{j\xi} + H_\xi z^+_{j\xi} - z^-_{j\xi})$, since $-\lambda^i_\xi q_\xi h_\xi z_\xi + \mu^i_\xi \chi_\xi z_\xi = -\lambda^i_\xi q_\xi h_\xi (z^+_\xi - z^-_\xi) + \mu^i_\xi (H_\xi z^+_\xi - z^-_\xi) \leq -\lambda^i_\xi q_\xi (H_\xi z^+_\xi - z^-_\xi) + \mu^i_\xi (H_\xi z^+_\xi - z^-_\xi)$. The Euler condition on φ_ξ implies that $-\lambda^i_\xi q_\xi + \mu^i_\xi = -\sum_{\eta \in \xi^+} \lambda^i_\eta (p_\eta B_\eta + q_\eta) < 0$. Hence $\mathcal{A}^i_\xi \leq 0$.

In case II: For any (φ, θ, ψ) satisfying budget and box constraints, with $(\theta, \psi) \geq 0$, and $|q_{j\xi}\varphi^i_{j\xi}| \leq M_j$ if $i \in \mathbf{D}$, the analogous dealer's the condition is $\limsup_{T\to\infty} \sum_{\xi:t(\xi)=T} \mathcal{B}^i_\xi \leq 0$, where $\mathcal{B}^i_\xi \equiv -\lambda^i_\xi q_\xi(\varphi_\xi + h_\xi \theta_\xi - \psi_\xi) + \mu^i_\xi(\varphi_\xi + \theta_\xi - \psi_\xi)$, and the corresponding non-dealer's condition is $\limsup_{T\to\infty} \sum_{\xi:t(\xi)=T} \mathcal{C}^i_\xi \leq 0$, where $\mathcal{C}^i_\xi \equiv -\lambda^i_\xi q_\xi(\varphi_\xi + \theta_\xi - h_\xi \psi_\xi) + \mu^i_\xi(\varphi_\xi + \theta_\xi - \psi_\xi)$.

Let us start with the non-dealer's condition. We have $\mathcal{C}^i_\xi \leq (-\lambda^i_\xi q_\xi + \mu^i_\xi)(\varphi_{j\xi} + \theta_\xi - \psi_\xi)$ since $(\lambda^i_\xi q_\xi h_\xi - \mu^i_\xi)\psi_\xi \leq (\lambda^i_\xi q_\xi - \mu^i_\xi)\psi_\xi$. Now, $-\lambda^i_\xi q_\xi + \mu^i_\xi < 0$ (again this follows from the Euler condition in φ_ξ), so $\mathcal{C}^i_\xi \leq 0$ and the desired condition holds.

Let us now look at the dealers' condition. Notice that $-\lambda^i_\xi + \frac{\mu^i_{j\xi}}{q_{j\xi}} \leq -\lambda^i_\xi h_{j\xi} + \frac{\mu^i_{j\xi}}{q_{j\xi}} \leq -\sum_{\eta\in\xi^+} \lambda^i_\xi r_{1j\xi} h_{j\xi}$ (from dealer's Euler condition in $\theta_{j\xi}$). Then, $\mathcal{B}^i_\xi \leq \sum_j (-\lambda^i_\xi + \frac{\mu^i_{j\xi}}{q_{j\xi}}) q_{j\xi}(\varphi_{j\xi} - \psi_{j\xi}) \leq \sum_j (\lambda^i_\xi - \frac{\mu^i_{j\xi}}{q_{j\xi}}) q_{j\xi}\theta_{j\xi}$. Now, for the dealers, the real value of long repo positions is uniformly bounded, that is, $q_{j\xi}\theta_{j\xi}/\sum_l p_{l\xi}$ is uniformly bounded. As $\lambda^i_\xi h_{j\xi} - \frac{\mu^i_{j\xi}}{q_{j\xi}} \geq 0$, it suffices to show that $\sum_{\xi:t(\xi)=t} \lambda_\xi \sum_l p_{l\xi} \in l^1$. We know that $\sum_{\xi\in D} \lambda_\xi p_\xi \omega^i_\xi < \infty$. Since $\omega^i \ggg 0$, we have $\sum_{\xi:t(\xi)=t} \lambda^i_\xi \sum_l p_{l\xi} \in l^1$, as desired. ■

Proof of Theorem 1: An equilibrium for a finite horizon T economy is defined by requiring market clearing and individual optimality subject to these modified constraints. We denote it by $(p^T, q^T, r^T, \bar{a}^T)$ and $(\lambda^{iT}_\xi, \mu^{iT}_\xi)_\xi$ be an associated vector of Lagrange multipliers of budget and box constraints (respectively) of agent i. Notice that $p^T_\xi \gg 0$ (by monotonicity), we can normalize (p^T_ξ, q^T_ξ) in the simplex.

Lemma T1: *For each node ξ we have $\lambda^{iT}_\xi \sum_l p^T_{l\xi} \leq U^i(\Omega)/\underline{\omega}^i_\xi$, where $\underline{\omega}^i_\xi = \min_l \omega^i_{l\xi}$.*

Proof: By the saddle point property, for any $(a_\xi)_{\xi\in D^T(\xi_0)}$ satisfying relevant sign constraints, we have

$$\sum_{\xi:t(\xi)\leq T} L^i_\xi(a_\xi, a_{\xi^-}; \lambda^{iT}_\xi, \mu^{iT}_\xi, p^T, q^T, r^T) \leq U^{iT}(\bar{x}^{iT}) \tag{30}$$

$$\lambda^{iT}_\xi g^{iT}_\xi(\bar{a}_\xi, \bar{a}_{\xi^-}; p^T, q^T, r^T) = 0, \quad \mu^{iT}_\xi f^{iT}_\xi(\bar{a}_\xi) = 0. \tag{31}$$

Now let a_ξ be the plan consisting of consumption ω^i_ξ, null security position and null repo positions for nodes ξ such that $t(\xi) \leq t-1$ while being zero at other nodes. Then the above saddle point inequality implies $\sum_{\xi:t(\xi)=t} \lambda^{iT}_\xi p^T_\xi \omega^i_\xi \leq U^{iT}(\bar{x}^{iT}) \leq U^i(\Omega)$ and we get the result. ■

Lemma T2: $\sum_l p^T_{l\xi_0}$ *is bounded away from zero.*

Proof: We consider the following plan for case I: $(\tilde{x}^T, \tilde{\varphi}^T, \tilde{z}^T) = ((1-p^T_{l\xi_0})x^{iT} + bE_{l\xi_0}, (1-p^T_{l\xi_0})\varphi^{iT}, (1-p^T_{l\xi_0})z^{iT})$, where $E_{l\xi_0}$ is the canonical vector in the direction of commodity l at node ξ_0. This plan satisfies the date 0 budget constraint as long as $p^T_{l\xi_0} b \leq p^T_{l\xi_0}(p^T_{\xi_0} x^{iT}_{\xi_0} + q^T_{\xi_0}(\varphi^{iT}_{\xi_0} + h_{\xi_0} z^{iT}_{\xi_0}) = p^T_{\xi_0}\omega^i_{\xi_0} + q^T_{\xi_0} e^i$. Observe that it is enough to make $b \leq \min_{l,j}\{\omega^i_{l\xi_0}, e^i_j\}$, as $(p^T_{\xi_0}, q^T_{\xi_0})$ belongs to the simplex. Subsequent budgent constraints are clearly satisfied too. This plan clearly satisfies box constraints. Now, if $p^T_{l\xi_0} \to 0$ then $(1-p^T_{l\xi_0})x^{iT}$ would converge to x^{iT} in the l^∞ norm topology, for which the utility is continuous. Thus, $U^{iT}(\tilde{x}^T) > U^{iT}(x^{iT})$ for T large enough, a contradiction. The proof for case II is analogous. ■

Lemma T3: *For each node ξ we have the sequence $(\lambda^{iT}_\xi)_T$ bounded.*

Proof: Lemmas T1 and T2 imply that $(\lambda^{iT}_{\xi_0})_T$ is bounded. To see what happens at the immediately following nodes, we recall that the first order condition on $\varphi_{j\xi}$ for the truncated economy is

$$\lambda^{iT}_\xi q^T_{j\xi} = \sum_{\eta \in \xi^+} \lambda^{iT}_\eta (q^T_{j\eta} + p^T_\eta B_{j\eta}) + \mu^{iT}_{j\xi}$$

Take $\xi = \xi_0$ and suppose that $\lambda^{iT}_\eta \to \infty$ for some $\eta \in \xi^+_0$. Then, $q^T_{j\eta}$ should go to 0 for every j. But, as (p^T_η, q^T_η) is in the simplex, is would follow that $\sum_l p^T_{l\eta} \to 1$. Then, by Lemma T1, $\lambda^{iT}_\eta \nrightarrow \infty$, a contradiction. We have shown that $(\lambda^{iT}_\xi)_T$ bounded for every ξ such that $t(\xi) = t(\xi_0) + 1$. The proof for the following nodes proceeds in the same way using the above first order condition. ■

Lemma T4: *For each node ξ we have the sequence $(\mu^{iT}_{j\xi})_T$ bounded.*

Proof: Recall that in case I the first order condition on $z_{j\xi}$ for the truncated economy is

$$\lambda^{iT}_\xi q^T_{j\xi} h_{j\xi} = \sum_{\eta \in \xi^+} \lambda^{iT}_\eta q^T_{j\xi} h_{j\xi} r_{j\xi} + \mu^{iT}_{j\xi} \chi_{j\xi}$$

where $\chi_{j\xi} = H_{j\xi}$ if $z^{iT}_{j\xi} > 0$ and $\chi_{j\xi} = 1$ otherwise. So, $\mu^{iT}_{j\xi} \leq \lambda^{iT}_\xi q^T_{j\xi} h_{j\xi}/H_{j\xi}$ and the result follows by Lemma T3.

In case II the first order condition on $\theta_{j\xi}$ for the truncated economy requires

$$\begin{aligned} \lambda_\xi^{iT} q_{j\xi}^T h_{j\xi} &\geq \sum_{\eta\in\xi^+} \lambda_\eta^{iT} q_{j\xi}^T h_{j\xi} r_{1j\xi} + \mu_{j\xi}^{iT} \text{ if } i \in \mathbf{D} \\ \lambda_\xi^{iT} q_{j\xi}^T &\geq \sum_{\eta\in\xi^+} \lambda_\eta^{iT} q_{j\xi}^T r_{2j\xi} + \mu_{j\xi}^{iT} \text{ if } i \in \mathbf{ND} \end{aligned}$$

For both types of agents, we have $\mu_{j\xi}^{iT} \leq \lambda_\xi^{iT} q_{j\xi}^T$, and Lemma T3 applies again. ■

We already know that as (p_ξ^T, q_ξ^T) is in the simplex, the sequence $(p_\xi^T, q_\xi^T)_T$ is bounded, node by node. For the price variable r_ξ^T that was left outside the simplex, we say the following:

Lemma T5: *The sequence $(r_\xi^T)_T$ is bounded for each ξ.*

Proof: We show that $R_\xi^T \equiv 1/r_\xi^T$ is bounded away from 0. In case I, the first order condition on $z_{j\xi}$ implies that

$$R_{j\xi}^T \geq \sum_{\eta\in\xi^+} \frac{\lambda_\eta^{iT}}{\lambda_\xi^{iT}}$$

So, if (for a subsequence) $R_{j\xi}^T \to 0$, then $\lambda_\eta^{iT}/\lambda_\xi^{iT} \to 0$, for any $\eta \in \xi^+$. Now, by (A1) we have that $\frac{\lambda_\eta^{iT}}{\lambda_\xi^{iT}} = \frac{\mathcal{D}_l u_\eta^i(x_\eta^{iT}) p_{l\xi}}{\mathcal{D}_l u_\xi^i(x_\xi^{iT}) p_{l\eta}}$. As $\frac{\mathcal{D}_l u_\eta^i(x_\eta^{iT})}{\mathcal{D}_l u_\xi^i(x_\xi^{iT})}$ is bounded away from zero, it follows that $\frac{p_{l\xi}}{p_{l\eta}} \to 0$, so $p_{l\xi} \to 0$ (as $p_{l\eta}$ is bounded). This is impossible for $\xi = \xi_0$, by Lemma T2.

For $\xi \neq \xi_0$ the following argument applies. Since $\frac{\mathcal{D}_l u_{\xi_0}^i(x_{\xi_0}^{iT})}{\mathcal{D}_l u_\xi^i(x_\xi^{iT})} = \frac{\lambda_{\xi_0}^{iT} p_{l\xi_0}}{\lambda_\xi^{iT} p_{l\xi}}$, where the left hand side is bounded (by A1), if $p_{l\xi} \to 0$, then $\frac{\lambda_{\xi_0}^{iT}}{\lambda_\xi^{iT}} \to 0$. Now, λ_ξ^{iT} is bounded, so $\lambda_{\xi_0}^{iT} \to 0$, implying $\mathcal{D}_l u_{\xi_0}^i(x_{\xi_0}^{iT}) \to 0$. However, $\liminf_T \mathcal{D}_l u_{\xi_0}^i(x_{\xi_0}^{iT}) \in \mathcal{D}_l u_{\xi_0}^i([0, \Omega_{\xi_0}])$, which by (A1) cannot contain 0.

The proof for case II is analogous, using the first order condition on $\theta_{j\xi}$. ■

Hence, the above Lemmas allow us to find for $(p^T, q^T, r^T, \bar{a}^T, (\lambda_\xi^{iT}, \mu_\xi^{iT})_{i,\xi})$ a cluster point $(p, q, r, \bar{a}, (\lambda_\xi^i, \mu_\xi^i)_{i,\xi})$, as $T \to \infty$ (for the countable product topology). We claim that $(p, q, r, \bar{a})$ is an equilibrium.

Notice that Euler conditions hold at $(p, q, r, \bar{a}, (\lambda_\xi^i, \mu_\xi^i)_{i,\xi})$, by taking limits on first order conditions of finite horizon economies. Then, by Propositions 1, 2 and 3 it suffices to show that the transversality condition is satisfied. This follows as in item (ii) of the proof of Proposition 1 since the following inequality holds for any T :

$$-\sum_{\xi:t(\xi)=t}\lambda_\xi^{iT}\cdot\mathcal{D}_2 g_\xi^i(p^T,q^T,r^T)\cdot\bar{a}_{\xi^-}^i+\sum_{\xi:T\geq t(\xi)>t}\lambda_\xi^{iT}p_\xi^T\omega_\xi^i\leq\sum_{\xi:t(\xi)\geq t}u_\xi^i(\bar{x}_\xi^i)$$

and we make $T\to\infty$ noticing that $L^i_{2\xi}(a_\xi,a_{\xi^-})=-\lambda_\xi^i\cdot\mathcal{D}_2 g_\xi^i(p,q,r)$. This cocludes the proof of Theorem 1.■

Lemma 4: *For the sequence of horizon T truncated optimization problems with a common price vector (p,q,r) the corresponding sequence of multipliers $(\lambda_\xi^{iT},(\mu_{j\xi}^{iT})_j)_T$ is bounded for each node ξ.*

Proof: This follows by adapting Lemmas T1 and T4 replacing (p^T,q^T,r^T) by (p,q,r). ■

Proof of Proposition 5: From equations (7.1) and (7.2) we have that

$$\sum_{\eta>\xi:t(\eta)=T}q_{j\eta}h_{j\eta}\lambda_\eta^i\left(1-\frac{1+\rho_\eta}{1+\iota_\eta}\right)\leq\sum_{\eta>\xi:t(\eta)=T}\mu_{j\eta}^i$$

and, therefore, letting $\underline{\mathrm{h}}_T=\min\limits_{\eta>\xi:t(\eta)=T}h_{j\eta}$ we have

$$\underline{\mathrm{h}}_T(1-\max_{\eta>\xi:t(\eta)=T}\frac{1+\rho_\eta}{1+\iota_\eta})\sum_{\eta>\xi:t(\eta)=T}q_{j\eta}\lambda_\eta^i\leq\sum_{\eta>\xi:t(\eta)=T}\mu_{j\eta}^i$$

Now, if the right hand side tends to zero, as $T\to\infty$, we have the claimed result, as $\sum_{\eta>\xi:t(\eta)=T}q_{j\eta}\lambda_\eta^i\nrightarrow 0$ implies $\max\limits_{\eta>\xi:t(\eta)=T}\frac{1+\rho_\eta}{1+\iota_\eta}\to 1$. To see that $\sum_{\eta>\xi:t(\eta)=T}\mu_{j\eta}^i\to 0$, we use inequality (24) and make $a_\eta=(0,e^i,0)$ for every node η. Then,

$$\sum_{\eta:t(\eta)\leq T}\lambda_\eta^i p_\eta(\omega_\eta^i+B_\eta e^i)+e^i\sum_{\eta:t(\eta)\leq T}\mu_{j\eta}^i\leq U^i(\bar{x}^i)$$

Making $T\to\infty$ we see that $\sum_{\eta:t(\eta)\leq T}\mu_{j\eta}^i$ converges.■

Proof of Proposition 6: It is easy to see that the Euler equation in φ_ξ is satisfied. Then, the Euler equation in z_ξ is satisfied if the repo rate would be decreased by

$$\varepsilon=\frac{\mu_\xi^i\left(p_{t(\xi)+1}B_{t(\xi)+1}+1\right)}{\sum_{\eta>\xi}\mu_\eta^i}\left(\frac{\chi_\xi}{h_\xi}-1\right)$$

But, in all events we get $\varepsilon=0$ as can be seen by the way χ_ξ and μ_ξ^i were defined.

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